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A nonlinear free boundary problem with a self-driven Bernoulli condition

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ABSTRACT. We study a Bernoulli type free boundary problem with two phases

$$J[u]=\int_\Omega |\nabla u(x)|^2\,dx+\Phi\big(\mathcal{M}_-(u),\mathcal{M}_+(u)\big),\quad u-\bar u\in W^{1,2}_0(\Omega),$$

where $\bar u\in W^{1,2}(\Omega)$ is a given boundary datum. Here, $\mathcal{M}_1$ and $\mathcal{M}_2$ are weighted volumes of $\{u\leqslant 0\}\cap\Omega$ and $\{u>0\}\cap\Omega$, respectively, and Φ is a nonnegative function of two real variables.

We show that, for this problem, the Bernoulli constant, which determines the gradient jump condition across the free boundary, is of global type and it is indeed determined by the weighted volumes of the phases.

In particular, the Bernoulli condition that we obtain can be seen as a pressure prescription in terms of the volume of the two phases of the minimizer itself (and therefore it depends on the minimizer itself and not only on the structural constants of the problem).

Another property of this type of problems is that the minimizer in Ω is not necessarily a minimizer in a smaller subdomain, due to the nonlinear structure of the problem.

Due to these features, this problem is highly unstable as opposed to the classical case studied by Alt, Caffarelli and Friedman. It also interpolates the classical case, in the sense that the blow-up limits of u are minimizers of the Alt-Caffarelli-Friedman functional. Namely, the energy of the problem somehow linearizes in the blow-up limit.

As a special case, we can deal with the energy levels generated by the volume term $\Phi(0,r_2)=r_2^{\frac{n-1}{n}}$, which interpolates the Athanasopoulos-Caffarelli-Kenig-Salsa energy, thanks to the isoperimetric inequality.

In particular, we develop a detailed optimal regularity theory for the minimizers and for their free boundaries.

CONTENTS

1. INTRODUCTION

After [1, 2], a classical problem in the free boundary theory consists in studying the minimizers of an energy functional which is the *linear superposition* of a Dirichlet energy and a volume term. In this case, minimizers are proved to be harmonic away from the free boundary. Also, minimizers naturally enjoy a free boundary condition which can be seen as a balance of the normal derivatives across the interface.

This type of problems has a natural interpretation in terms of two dimensional flows of two irrotational, incompressible and inviscid fluids. Indeed, if the fluids have velocities $\mathbf{v}^\pm=\nabla\phi^\pm$, for some potential functions $\phi^\pm$, it holds that $\Delta\phi^\pm=0$ whenever $\phi^\pm\neq 0$. In addition, the Bernoulli law states that

$$\frac{p^\pm(x)}{\rho^\pm}=\frac{|\mathbf{v}^\pm|}{2}+C_\pm \tag{1.1}$$

along every streamline, i.e. lines for which the tangent is in the direction of the velocity (in other words the level sets of the harmonic conjugate $\psi^\pm$ of $\phi^\pm$). Here $C_\pm$ are constants depending on the streamline and $p^\pm$ is the pressure from either side. If the free boundary is smooth then the pressure $p^\pm$ is continuous and therefore from Cauchy-Riemann equations, after normalization, we get that (assuming that the densities $\rho^\pm$ are constant)

$$p^\pm(x) + \frac{|\nabla\psi^\pm|^2}{2} = C_\pm \quad \Longrightarrow \quad |\nabla\psi^+|^2 - |\nabla\psi^-|^2 = 2(C_+ - C_-). \tag{1.2}$$

In this interpretation, we see that the free boundary condition in [1, 2] is a variational version of the classical Bernoulli law (and in fact it justifies the validity of a weak version of this law at points where the free boundary is not regular).

In this paper, we consider the case in which the energy functional is a *nonlinear superposition* of a Dirichlet energy and a volume term.

We will show that general nonlinearities may produce pathologic examples, in which minimizers may not exist, or in which the free boundary of the minimizers is not smooth. Nevertheless, under suitable structural assumptions on the nonlinearity, we will show that a sufficiently strong existence and regularity theory holds true.

In addition, we will obtain a new version of the free boundary condition, which, in our case, turns out to be of "global" type. As a matter of fact, in our case, the free boundary condition may still be seen as a balance between the normal derivatives from the two sides of the free boundary, but, differently from the classical case, this balance changes from point to point of the free boundary and the change takes into account quantities that are defined globally, and not only locally (e.g., they include the nonlinearity itself and the weighted volumes of the phases of the minimizers).

Roughly speaking, in this new free boundary condition, the quantities $C_\pm$ in (1.1) are not constant anymore and they are not locally determined. In other words, they depend not only on the streamline but also on the weighted volumes that the streamline separates, and, above all, on the minimizers themselves: for this reason, we named this type of condition *self-driven*.

An explicit geometric example related to our problem can be given in terms of the isoperimetric inequality

$$\frac{Area(\partial\Omega^+)}{\left[\mathcal{L}^n(\Omega^+)\right]^{1-\frac{1}{n}}} \geqslant \frac{Area(\mathbb{S}^{n-1})}{\left[\mathcal{L}^n(B_1)\right]^{1-\frac{1}{n}}} =: c_n,$$

that is

$$Area(\partial\Omega^+) \geqslant c_n \left[\mathcal{L}^n(\Omega^+)\right]^{1-\frac{1}{n}}.$$

Consequently,

$$J_{\mathrm{ACKS}}[u] := \int_\Omega |\nabla u|^2 + Area(\partial\Omega^+) \geqslant \int_\Omega |\nabla u|^2 + c_n \left[\mathcal{L}^n(\Omega^+)\right]^{1-\frac{1}{n}} =: J[u]. \tag{1.3}$$

In this sense, the energy functional J_{ACKS} studied in [4] provides an upper bound for the energy functional J. Notice that J_{ACKS} is a *linear interpolation* of energies (the second one being an area), while J is a *nonlinear interpolation* of energies (the second one being of volume type, but scaling like an area). The functional J in (1.3) is indeed a particular case of the ones that we study in the present paper.

We observe that this type of problems is related to the Ginzburg-Landau model with three competing rates which balance each other for a suitable choice of the structural parameter. The exact choice of the rate gives, in the limit, the energy J_{ACKS}. Thus, in this spirit, the functional J in (1.3) describes a model in which the equilibrium is reached in terms of the best approximation of isoperimetric inequality under given constraints.

In the following subsections, we will describe the formal mathematical setting of the problem, the main results and the organization of this paper.

1.1. **Problem set-up.** Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set. In what follows, $\lambda_1 \geqslant 0$ and $\lambda_2 > 0$ are given constants. For a given measurable function $Q : \Omega \to \mathbb{R}$, bounded by two positive numbers

$$0 < Q_1 \leqslant Q(x) \leqslant Q_2 < \infty, \tag{1.4}$$

we define the weighted partial volumes

$$\mathcal{M}_1(u) := \lambda_1 \int_\Omega Q(x)\,\chi_{\{u\leqslant 0\}}(x)\,dx \quad \text{and} \quad \mathcal{M}_2(u) := \lambda_2 \int_\Omega Q(x)\,\chi_{\{u>0\}}(x)\,dx$$

and the total volume

$$\lambda_\Omega := \int_\Omega Q(x)\,dx.$$

It is easy to see that

$$\mathcal{M}_1(u) = \lambda_1 \left(\lambda_\Omega - \int_\Omega Q(x)\,\chi_{\{u>0\}}(x)\,dx\right) = \lambda_1\big(\lambda_\Omega - \lambda_2^{-1}\mathcal{M}_2(u)\big). \tag{1.5}$$

Let also $\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$. For a given $\Phi \in C^0\big(\overline{\mathbb{R}_+} \times \overline{\mathbb{R}_+},\, \overline{\mathbb{R}_+}\big) \cap C^1\big(\mathbb{R}_+ \times \mathbb{R}_+,\, \mathbb{R}_+\big)$ such that $\Phi(0,0) = 0$, we consider

$$\Phi_0(r) := \Phi\Big(\lambda_1\big(\lambda_\Omega - \lambda_2^{-1} r\big),\, r\Big) \tag{1.6}$$

and suppose that

$$\Phi_0'(r) \geqslant 0 \text{ for any } r \in (0,\, \lambda_2\,\lambda_\Omega). \tag{1.7}$$

In view of (1.5),

$$\Phi\big(\mathcal{M}_1(u), \mathcal{M}_2(u)\big) = \Phi_0\big(\mathcal{M}_2(u)\big).$$

In this paper we study the minimization problem of the energy functional

$$\begin{aligned} J[u] &:= \int_\Omega |\nabla u(x)|^2\,dx + \Phi\big(\mathcal{M}_1(u), \mathcal{M}_2(u)\big) \\ &= \int_\Omega |\nabla u(x)|^2\,dx + \Phi_0\big(\mathcal{M}_2(u)\big) \end{aligned} \tag{1.8}$$

in the admissible class

$$\mathcal{A} := \{u \in W^{1,2}(\Omega), \text{ with } u - \bar u \in W_0^{1,2}(\Omega)\}, \tag{1.9}$$

where $\bar u \in W^{1,2}(\Omega)$.

For a given minimizer u of (1.8) the free boundary of u is denoted by $\Gamma := \partial\Omega^+(u)$, where

$$\Omega^+(u) := \{x \in \Omega \text{ s.t. } u(x) > 0\}. \tag{1.10}$$

This problem can be viewed as an extrapolation of the classical free boundary problem of Alt and Caffarelli [1], where the authors studied the local minimizers of the energy

$$J_{\mathsf{AC}}[u] := \int_\Omega \Big(|\nabla u(x)|^2 + Q(x)\,\chi_{\{u>0\}}(x)\Big)\,dx. \tag{1.11}$$

Indeed, the functional in (1.8) reduces to that in (1.11) with the choices $\lambda_1 := 0$, $\lambda_2 := 1$ and $\Phi(r_1, r_2) := r_2$.

More generally, the functional in (1.8) is also an extrapolation of the two-phase free boundary problem in [2], in which, instead of the functional in (1.8), the minimization problem dealt with the energy

$$J_{\mathsf{ACF}}[u] := \int_\Omega \Big(|\nabla u(x)|^2 + \lambda_1\,Q(x)\,\chi_{\{u\leqslant 0\}}(x) + \lambda_2\,Q(x)\,\chi_{\{u>0\}}(x)\Big)\,dx, \tag{1.12}$$

since the functional in (1.8) reduces to that in (1.12) with the choice $\Phi(r_1, r_2) := r_1 + r_2$ (in this case, condition (1.7) reduces to $\lambda_2 \geqslant \lambda_1$, compare with the assumptions of Theorem 2.3 in [2]).

In this sense, the energy functional in (1.8) provides a free boundary problem that is either one-phase (when $\lambda_1 = 0$) or two-phase (when $\lambda_1 \neq 0$) and in which the interfacial energy depends on the volume of the phases in a possibly nonlinear way, which is described by the function Φ. The principal difference from the classical case is that the free boundary condition is determined by the weighted volumes of the phases and hence its Bernoulli constant is of global type and varies from one minimizer to another.

More precisely, if the free boundary $\Gamma := \partial\{u > 0\}$ is a smooth hypersurface then

$$|\nabla u^+(p)|^2 - |\nabla u^-(p)|^2 = \Lambda(p), \quad p \in \Gamma, \tag{1.13}$$

where

(1.14)

$$\Lambda(p) := \left[\lambda_2 \partial_{r_2}\Phi\left(\lambda_1 \int_\Omega Q\,\chi_{\{u<0\}},\; \lambda_2 \int_\Omega Q\,\chi_{\{u>0\}}\right) - \lambda_1 \partial_{r_1}\Phi\left(\lambda_1 \int_\Omega Q\,\chi_{\{u<0\}},\; \lambda_2 \int_\Omega Q\,\chi_{\{u>0\}}\right)\right] Q(p).$$

In the classical case Λ is a prescribed function and only depends on the ambient space at a given point. Conversely, since in our setting Λ depends in a nonlinear fashion on global quantities such as $\mathcal{M}_1$ and $\mathcal{M}_2$, which in turn depend on the solution, it is natural to expect that the problem is going to be highly unstable (other global free boundary conditions arise in unstable free boundary problems as the one dealt with in formula (1.12) in [10]). In particular, comparing (1.2) with (1.13) and (1.14), we may consider the free boundary condition of our problem as a nonlinear prescription of the pressure in terms of the volume of the two phases of the minimizer.

In our framework, the instability produced by the nonlinear superposition of energies may be, in general, quite severe, and, in fact, *the minimizers do not always exist*, as we will see in Section 3. Thus some structural assumptions are needed in order to develop a meaningful theory.

Interesting examples of nonlinearities that we can take into account are given by $\Phi(r_1, r_2) := r_2^{\frac{n-1}{n}}$ and $\Phi(r_1, r_2) := (r_1 + r_2)^{\frac{n-1}{n}}$. We notice that this type of nnonlinearities provides a scaling which is naturally induced by the isoperimetric inequality. For instance, the minimizers of the Athanasopoulos-Caffarelli-Kenig-Salsa functional [4]

$$\int_\Omega |\nabla u|^2 + \mathrm{Per}_\Omega(\{u > 0\})$$

generate energy levels that are above the ones of our functional. If the phases have isoperimetric property then this levels coincide. In this sense, the main difference between the energy functionals in (1.8) and (1.11)–(1.12) lies in the different scaling of the volume term. This can be seen, as a paradigmatic example, by looking at the functional

$$\int_\Omega |\nabla u|^2 + (\mathcal{L}^n(\{u > 0\} \cap \Omega))^{\frac{n-1}{n}} .$$

We remark that different scalings in perimeter/volume terms may cause instability phenomena in the corresponding minimization arguments, namely a minimizer in a given domain is not necessarily a minimizer in a smaller domain, see [10].

Other cases of interest for the nonlinearity are the following ones:

- Φ only depends on the sum of the masses of the two phases, namely when $\Phi(r_1, r_2) = \tilde{\Phi}(r_1 + r_2)$ (notice that in this case, condition (1.7) is implied by the two conditions $\lambda_2 \geqslant \lambda_1$ and $\tilde{\Phi}'(r) \geqslant 0$ for any $r > 0$).
- Φ only depends on the the sum of different powers of the masses of the two phases, namely

  $$\Phi(r_1, r_2) := \frac{r_1^{1+\alpha}}{1+\alpha} + \frac{r_2^{1-\beta}}{1-\beta},$$

  with $\alpha \geqslant 0$ and $\beta > 0$. Notice that in this case condition (1.7) is satisfied if λ_1 is sufficiently small (possibly in dependence of λ_2 and λ_Ω).

1.2. **Main results.** The main results of this paper deal with the regularity of the minimizers and of their free boundary. We stress that, in general, minimizers *may not exist* and, when minimizers exist, their free boundary *may be irregular*. We will present some explicit examples of these pathologies in Sections 3 and 4.

In spite of these examples, under suitable structural assumptions, a good regularity theory holds true.

To this aim, the assumption that we take is that

$$\text{(1.7) holds true and that } \Phi_0 \text{ is concave.} \tag{1.15}$$

Recalling the notation in (1.10), we will also suppose that, for a given minimizer u,

$$\mathcal{L}^n\big(\Omega^+(u)\big) \geqslant \varpi > 0. \tag{1.16}$$

We will see that this assumption is not restrictive and it is satisfied in all nontrivial cases (a precise statement will be given in Lemma 5.1). Then, our first result deals with the regularity of the gradient of the minimizers in BMO spaces, and it goes as follows:

Theorem 1.1. *Let u be a minimizer in Ω for the functional J in* (1.8)*. Assume that* (1.15) *and* (1.16) *hold true.*

Then $\nabla u \in BMO_{\rm loc}(\Omega, \mathbb{R}^n)$. More precisely, for any $D \Subset \Omega$, there exists $C > 0$, possibly depending on ϖ, Q, Ω and D, such that

$$\sup_{B_r(x_0)\subseteq D} \fint_{B_r(x_0)} |\nabla u(x) - (\nabla u)_{x_0,r}|\, dx \leqslant C.$$

As a consequence of Theorem 1.1, we also obtain the following result:

Corollary 1.2. *Let u be a minimizer in Ω for the functional J in* (1.8)*. Assume that* (1.15) *and* (1.16) *hold true. Then:*

- *u is locally log-Lipschitz continuous, namely it is continuous, with modulus of continuity bounded by $\sigma(t) = t|\log t|$.*
- *u is harmonic in the set $\Omega^+(u)$.*
- *For any $D \Subset \Omega$, there exists $C > 0$, possibly depending on ϖ, Q, Ω and D, such that*

$$\left| \frac{1}{r^{n-1}} \int_{\partial B_r(x_0)} u \right| \leqslant C\, r, \tag{1.17}$$

for any $x_0 \in \Gamma$, as long as $B_r(x_0) \subseteq D$.

The regularity of the minimizers can also be improved, to reach the optimal Lipschitz regularity, as given by the following result:

Theorem 1.3. *Let u be a minimizer in Ω for the functional J in* (1.8)*. Assume that* (1.15) *and* (1.16) *hold true. Then $u \in C^{0,1}_{loc}(\Omega)$.*

We also deal with the geometric properties of the minimizers, obtaining optimal quantitative results. In particular, we prove nondegeneracy of minimizers and linear growth from the free boundary, as stated in the next result:

Theorem 1.4. *Let u be a minimizer in Ω of the energy functional J in* (1.8)*, Ω_0^+ a connected component of the positivity set $\Omega^+(u)$, and $x_0 \in \partial\Omega_0^+$.*

Assume that (1.15) *and* (1.16) *hold true. Suppose that $r > 0$ is small enough such that $B_r(x_0) \Subset \Omega$ and*

$$\mathcal{L}^n\big(\Omega^+(u) \setminus B_r(x_0)\big) \geqslant \frac{\varpi}{2}. \tag{1.18}$$

Then, for any $\kappa \in (0,1)$ there exists a positive constant c, depending only on ϖ, κ and Q, such that if

$$\fint_{B_r(x_0)\cap\Omega_0^+} u^2 < cr^2,$$

then $u^+ = 0$ in $B_{\kappa r}(x_0) \cap \Omega_0^+$.

In particular, for any domain $D \Subset \Omega$ there exists a positive constant c, depending only on ϖ, Q, Ω and dist$(D, \partial\Omega)$, such that

$$\fint_{B_r(x_0)\cap\Omega_0^+} u^2 \geqslant cr^2, \tag{1.19}$$

for any $x_0 \in \partial\Omega_0^+ \cap D$ and $r > 0$, such that $B_r(x_0) \Subset D$.

Interestingly, the result in Theorem 1.4 holds true in any connected component of the positivity set of the minimizers.

In this paper, we also establish several density results for minimizers, that can be of independent interest as well, and that can be used to establish the minimizing properties of the blow-up limits of the minimizers, which indeed turn out to be minimizers of more classical free boundary problems. In this setting, the result that we obtain is the following:

Theorem 1.5. *Let u be a minimizer in Ω for the functional in* (1.8) *and let u_0 be the blow-up limit*[1]*. Assume that Q is continuous at 0 and that* (1.15) *and* (1.16) *hold true.*

Then, for any fixed $r > 0$, we have that u_0 is a minimizer of the functional

$$J_0[w] := \int_{B_r(x_0)} |\nabla w|^2 + \lambda_0\, \mathcal{L}^n\big(B_r(x_0) \cap \{w > 0\}\big),$$

[1] A standard, explicit definition of the blow-up limit and the existence of such limit is given in Proposition 10.1.

where

$$\lambda_0 := \lambda_2 \, Q(0) \, \Phi_0' \left(\lambda_2 \int_{B_r(x_0)} Q(x) \, \chi_{\{u>0\}}(x) \, dx \right).$$

We also obtain partial regularity results (valid in any dimension) for free boundary points, as stated in the following result:

Theorem 1.6. *Let u be a minimizer in Ω for the functional J in* (1.8). *Assume that* (1.15) *and* (1.16) *are satisfied.*

Then, the following statements hold true:

(i) Δu^+ *is a Radon measure and, for any $x \in \Gamma$ and any $r > 0$ such that $B_{2r}(x) \subset \Omega$ and $\mathcal{L}^n\big(\Omega^+(u) \setminus B_r(x)\big) \geqslant \varpi/2$, we have that*

$$\int_{B_r(x)} \Delta u^+ \leqslant \frac{1}{r} \int_{B_{2r}(x)} |\nabla u^+|.$$

(ii) *For any subdomain $D \Subset \Omega$ there exists $r_0 > 0$ such that*

$$\int_{B_r(x)} \Delta u^+ \geqslant c r^{n-1},$$

for any $r \in (0, r_0)$, $x \in \Gamma$ and such that $B_r(x) \subseteq D$, for a suitable $c > 0$.

(iii) *If $B_r \subseteq \Omega$, then*

$$\mathcal{H}^{n-1}\big(B_r \cap \{u > 0\}\big) < +\infty \tag{1.20}$$

$$\text{and} \quad \mathcal{H}^{n-1}\big((B_r \cap \partial\{u > 0\}) \setminus \partial_{\mathrm{red}}\{u > 0\}\big) = 0. \tag{1.21}$$

In dimension 2, we also obtain a complete regularity theory for the minimizers. This result goes as follows:

Theorem 1.7. *Let $n = 2$ and assume that* (1.15) *and* (1.16) *are satisfied. Then each free boundary point is critically flat and hence $\partial\{u > 0\}$ is continuously differentiable.*

We stress that, in this paper, the techniques that we develop are strong enough to allow a unified treatment of the one and two phase cases simultaneously.

Furthermore, all the results of this paper are valid in any dimension (with the only exception of Lemmata 12.7 and 12.8, and Theorem 1.7).

1.3. **Organization of the paper.** The paper is organized as follows. In Section 2 we prove the existence of minimizers. The proof is standard and is based on a semicontinuity argument and on a refinement of Egoroff's theorem for Sobolev functions.

In Section 3, an explicit example of volumetric function Φ_0 for which no solution exists is constructed. While $\Phi_0(r)$ suffers a jump at $r = 1$, the non-existence is still surprising because it shows that, for such Φ_0, the set of admissible functions is not empty. In Section 4 we construct another explicit example of Φ_0 that does not satisfy the structural assumption in (1.7): in this case, minimizers do exist, but their free boundary is irregular.

That done, we begin to establish the basic properties of the minimizers in Section 5. First a lower bound for the positivity set $\Omega^+(u)$ is proved for a suitable boundary condition. Then we show that the gradients of minimizers are locally BMO functions, that is we prove Theorem 1.1. This, in turn, implies that u is locally log-Lipschitz continuous, as given by Corollary 1.2. For the one-phase problem this immediately implies the linear growth of u away from free boundary. One of the by-products of the local BMO estimate is the coherent growth estimate (1.17). Using this and the Alt-Caffarelli-Friedman monotonicity theorem we prove the local Lipschitz regularity for the minimizers of the two-phase problem, as stated in Theorem 1.3.

Then, we use a domain variation argument, to derive the nonlocal Bernoulli condition in Section 6.

The non-degeneracy of minimizers given by Theorem 1.4 is proved in Section 7.

In Section 8 we show that for every ball B centered at the free boundary there exists a smaller ball $B' \subset \Omega^+(u) \cap B$ such that $\mathcal{L}^n(B') \geqslant c\mathcal{L}^n(B)$, for some universal constant $c > 0$. Moreover, in Section 9 we show the slightly weaker statement that the non-positivity set $\{u \leqslant 0\}$ has locally uniformly positive Lebesgue density at the free boundary points.

Section 10 is devoted to the study of the properties of the blow-up limits and to the proof of Theorem 1.5. In particular, we show that the blow-up limits of the minimizers become global solutions for the Alt-Caffarelli-Friedman functional provided that Q is continuous.

In Section 11 we prove the partial regularity of the free boundary, as given by Theorem 1.6, namely, that the $\partial\{u > 0\}$ is of locally finite perimeter and the reduced boundary has full $\mathcal{H}^{n-1}$ measure in $\partial\{u > 0\}$. In particular, we show that the measure theoretic normal exists at $\mathcal{H}^{n-1}$ a.e. point of $\partial\{u > 0\}$.

In Section 12 we prove that at the flat free boundary points the free boundary is regular and establish the full regularity of the free boundary in two dimensions, as stated in Theorem 1.7.

Notation

Let us fix some notation.

- $\mathcal{L}^n$ is the n dimensional Lebesgue measure.
- $\mathcal{H}^{n-1}$ is the $n-1$ dimensional Hausdorff measure.
- $u^+(x) := \max\{u(x), 0\}$ and $u^-(x) := -\min\{u(x), 0\}$ are the positive and the negative parts of u, respectively, so that $u = u^+ - u^-$.
- $\lambda_1 \geqslant 0$ and $\lambda_2 > 0$ are given constants.
- $\Omega^+(u) := \{x \in \Omega : u(x) > 0\}$ and $\Omega^-(u) := \{x \in \Omega : u(x) < 0\}$ are the positivity and the negativity sets of u, respectively,
- $\varpi > 0$ is the constant providing a lower bound for the weighted volume of $\Omega^+(u)$.
- $\Gamma = \partial\{u > 0\}$ is the free boundary.
- The open balls are denoted by $B_r(x_0) := \{x \in \mathbb{R}^n$ s.t. $|x - x_0| < r\}$ and $B_r := B_r(0)$.
- $C^{0,1}_{loc}(\Omega)$ is the class of locally Lipschitz continuous functions in Ω.
- The mean value integral is $\fint_E f = \frac{1}{\mathcal{L}^n(E)} \int_E f$.
- Various universal constants are often denoted by C, for simplicity.

2. Existence and basic properties of minimizers

In this section we prove that there exists $u \in \mathcal{A}$ minimizing (1.8), where $\mathcal{A}$ is defined in (1.9). We also show that under the condition (1.7), u is globally subharmonic in Ω.

2.1. Existence of minimizers.

Lemma 2.1. *Fix $\bar{u} \in W^{1,2}(\Omega)$. Then, there exists $u \in \mathcal{A}$ such that*

$$J[u] \leqslant J[v]$$

for any $v \in \mathcal{A}$.

Proof. The proof is a standard lower semicontinuity argument (we give the details for the facility of the reader). We notice that

$$\mathcal{M}(u_0) \leqslant (\lambda_1 + \lambda_2)\, Q_2\, \mathcal{L}^n(\Omega) < +\infty$$

and so $J[\bar{u}] < +\infty$. Now, let $u_k \in \mathcal{A}$ be a minimizing sequence. We observe that $\bar{u} \in \mathcal{A}$, hence for sufficiently large k we may suppose that

$$J[u_k] \leqslant J[\bar{u}] < +\infty. \tag{2.1}$$

Set $v_k := u_k - \bar{u} \in W^{1,2}_0(\Omega)$. As a consequence of (2.1), we have that

$$\sup_{k \in \mathbb{N}} \|\nabla v_k\|_{L^2(\Omega, \mathbb{R}^n)} < +\infty,$$

and so, by Poincaré inequality, also

$$\sup_{k \in \mathbb{N}} \|v_k\|_{L^2(\Omega)} < +\infty.$$

Therefore, up to a subsequence, v_k converges weakly to some $v \in W_0^{1,2}(\Omega)$, strongly in $L^2(\Omega)$ and a.e. in Ω. Then, ∇v_k converges weakly to ∇v in $L^2(\Omega, \mathbb{R}^n)$. So, if we set $u := v + \bar{u}$, we have that $u \in \mathcal{A}$, $u_k \to u$ in $L^2(\Omega)$ and a.e. in Ω, and $\nabla u_k \to \nabla u$ weakly in $L^2(\Omega, \mathbb{R}^n)$. In particular,

$$\liminf_{k\to+\infty} \int_\Omega |\nabla u_k|^2 \geqslant \int_\Omega |\nabla u|^2. \tag{2.2}$$

Now we observe that

$$\liminf_{k\to+\infty} \int_\Omega \lambda_2 Q(x)\, \chi_{\{u_k>0\}}(x)\, dx \geqslant \int_{\Omega\setminus E_\varepsilon} \lambda_2 Q(x)\, \chi_{\{u>0\}}(x)\, dx. \tag{2.3}$$

For this, let $\varepsilon > 0$ be fixed. Using the refinement of Egoroff's theorem for $W^{1,2}$ functions, it follows that there exists a subset $E_\varepsilon \subset \Omega$ such that $u_k \to u$ uniformly in $\Omega \setminus E_\varepsilon$ and $\mathrm{cap}_2(E_\varepsilon) < \varepsilon$ where cap_2 is the 2-capacity. Thus

$$\liminf_{k\to+\infty} \int_\Omega \lambda_2 Q(x)\, \chi_{\{u_k>0\}}(x)\, dx \geqslant \int_{\Omega\setminus E_\varepsilon} \lambda_2 Q(x)\, \chi_{\{u>0\}}(x)\, dx - \lambda_2 Q_2 |E_\varepsilon|.$$

Sending $\varepsilon \to 0$ the result in (2.3) follows.

From (2.3), we deduce that

$$\liminf_{k\to+\infty} \mathcal{M}_2(u_k) = \liminf_{k\to+\infty} \int_\Omega \lambda_2 Q(x)\, \chi_{\{u_k>0\}}(x)\, dx \geqslant \int_\Omega \lambda_2 Q(x)\, \chi_{\{u>0\}}(x)\, dx = \mathcal{M}_2(u). \tag{2.4}$$

Furthermore, by (1.5),

$$0 \leqslant \mathcal{M}_1(u) = \lambda_1 \left(\lambda_\Omega - \lambda_2^{-1} \mathcal{M}_2(u)\right)$$

and therefore $\mathcal{M}_2(u) \in [0,\ \lambda_2\lambda_\Omega]$ and, similarly, $\mathcal{M}_2(u_k) \in [0,\ \lambda_2\lambda_\Omega]$.

Therefore, since, by (1.7), the function Φ_0 is nondecreasing in $[0,\ \lambda_2\lambda_\Omega]$, we deduce from (2.4) that

$$\liminf_{k\to+\infty} \Phi_0\big(\mathcal{M}_2(u_k)\big) = \Phi_0\left(\liminf_{k\to+\infty} \mathcal{M}_2(u_k)\right) \geqslant \Phi_0\big(\mathcal{M}_2(u)\big).$$

This and (2.2) give that

$$\liminf_{k\to+\infty} J[u_k] \geqslant J[u],$$

hence u is the desired minimizer. □

We remark that if $Q := 0$ and $\mathcal{A}^+ := \{u \in \mathcal{A} \text{ s.t. } u \geqslant 0\}$, then the functional J in (1.8) becomes $J[u] := \int_\Omega |\nabla u(x)|^2\, dx$, and so the minimizer of J in $\mathcal{A}^+$ is the solution to the Dirichlet problem.

This means that the free boundary $\partial\{u > 0\}$ arises only in the regions where $Q > 0$. Therefore, from now on we assume that (1.4) holds true.

2.2. **Euler-Lagrange equations.** We now state the basic properties of the minimizers. The starting point is to derive the differential inequalities that the minimizers satisfy in Ω.

Lemma 2.2. *Let u be a minimizer in Ω for the functional J in* (1.8) *and suppose that* (1.7) *holds true. Then u is subharmonic.*

Proof. We use some classical ideas in Lemma 2.2 of [1] and Theorem 2.3 in [2], combining them here with condition (1.7). For this, we consider a ball $B \Subset \Omega$ and the function v which is harmonic in B and coincides with u in $\Omega \setminus B$. We also take $w := \min\{u, v\}$. Then, w is an admissible competitor for u and therefore $J[u] \leqslant J[w]$, that is

$$I := \int_B |\nabla u(x)|^2\, dx - \int_B |\nabla w(x)|^2\, dx \leqslant \Phi_0\big(\mathcal{M}_2(w)\big) - \Phi_0\big(\mathcal{M}_2(u)\big). \tag{2.5}$$

On the other hand, if we set $z := \max\{u - v, 0\}$, we have that

$$\begin{aligned} I &= \int_B \big(\nabla(u-w)(x)\big) \cdot \big(\nabla(u+w)(x)\big)\, dx = \int_{B\cap\{u>v\}} \big(\nabla(u-v)(x)\big) \cdot \big(\nabla(u+v)(x)\big)\, dx \\ &= \int_{B\cap\{u>v\}} \big|\nabla(u-v)(x)\big|^2\, dx + 2\int_{B\cap\{u>v\}} \big(\nabla(u-v)(x)\big) \cdot \nabla v(x)\, dx \\ &= \int_{B\cap\{u>v\}} \big|\nabla(u-v)(x)\big|^2\, dx = \int_B |\nabla z(x)|^2\, dx. \end{aligned}$$

Inserting this into (2.5), we obtain that

$$\int_B |\nabla z(x)|^2\, dx \leqslant \Phi_0\big(\mathcal{M}_2(w)\big) - \Phi_0\big(\mathcal{M}_2(u)\big). \tag{2.6}$$

Moreover, we have that $w \leqslant u$ and therefore $\chi_{\{w>0\}} \leqslant \chi_{\{u>0\}}$. Accordingly, $\mathcal{M}_2(w) \leqslant \mathcal{M}_2(u)$ and then, in light of (1.7), we obtain that $\Phi_0\big(\mathcal{M}_2(w)\big) \leqslant \Phi_0\big(\mathcal{M}_2(u)\big)$. From this and (2.6) we deduce that z is constant in B. Since z vanishes in $\Omega \setminus B$, we conclude that z vanishes in Ω and therefore that $u \leqslant v$, which establishes the desired result. ☐

3. Non existence of minimizers for irregular nonlinearities

In this section, we observe that when the regularity and the structural assumptions on Φ_0 are violated, minimizers may not exist. To exhibit this phenomenon in an explicit and concrete example, we consider the case in which $\Omega := (0,1) \subset \mathbb{R}$, $\lambda_2 := Q := 1$, $\bar{u}(x) := x$ for any $x \in [0,1]$, and

$$\Phi(r_1, r_2) := \begin{cases} r_2 & \text{if } r_2 \in \big[0,\, \frac{1}{2}\big], \\ \frac{5-2r_2}{8} & \text{if } r_2 \in \big(\frac{1}{2},\, 1\big), \\ 1 & \text{if } r_2 \in [1, +\infty). \end{cases}$$

Notice that, with this setting,

$$\Phi_0(r) = \begin{cases} r & \text{if } r \in \big[0,\, \frac{1}{2}\big], \\ \frac{5-2r}{8} & \text{if } r \in \big(\frac{1}{2},\, 1\big), \\ 1 & \text{if } r \in [1, +\infty). \end{cases} \tag{3.1}$$

For this choice of Φ_0, *there exists no minimizer* u^* for the energy functional in (1.8) with the condition that $u^* - \bar{u} \in W_0^{1,2}(\Omega)$.

To see this, let us suppose, by contradiction, that such minimizer exists. Then,

$$\int_\Omega |\dot{u}^*|^2 \leqslant J[u^*] \leqslant J[\bar{u}] = 1 + \Phi_0(1) = 2. \tag{3.2}$$

As a consequence,

$$1 - 0 = u^*(1) - u^*(0) = \int_{(0,1)\cap\{u^*>0\}} \dot{u}^* \leqslant \sqrt{\int_\Omega |\dot{u}^*|^2}\, \sqrt{\mathcal{L}^1\big((0,1) \cap \{u^* > 0\}\big)} \leqslant \sqrt{2}\, \sqrt{\mathcal{M}_2(u^*)}$$

and so

$$\mathcal{M}_2(u^*) \geqslant \frac{1}{2}. \tag{3.3}$$

We claim that

$$\{u^* = 0\} \text{ has positive measure.} \tag{3.4}$$

To check this, we argue by contradiction and assume that $\mathcal{L}^1((0,1) \cap \{u^* = 0\}) = 0$, hence $\mathcal{M}_2(u^*) = 1$. Consequently, since $\bar{u}$ is a minimizer for the Dirichlet energy in $(0,1)$, we find that

$$J[u^*] = \int_0^1 |\dot{u}^*|^2 + 1 \geqslant \int_0^1 |\dot{\bar{u}}|^2 + 1 = 2. \tag{3.5}$$

Now we define, for any $\delta \in \big(0, \frac{1}{2}\big)$,

$$u_\delta(x) := \begin{cases} 0 & \text{if } x \in [0,\, \delta], \\ \frac{x-\delta}{1-\delta} & \text{if } x \in (\delta,\, 1]. \end{cases}$$

Then, there holds

$$J[u^*] \leqslant J[u_\delta] = \int_\delta^1 \left|\frac{1}{1-\delta}\right|^2 + \Phi_0(1-\delta) = \frac{1}{1-\delta} + \frac{5 - 2(1-\delta)}{8}.$$

Accordingly, by taking δ as small as we wish, we obtain

$$J[u^*] \leqslant 1 + \frac{3}{8}.$$

This inequality is in contradiction with (3.5) and so it proves (3.4).

In particular, from (3.4), we can take a Lebesgue point $p\in(0,1)$ for $\{u^*=0\}$. Thus, if $\varepsilon>0$ is sufficiently small, we have that

$$\mathcal{L}^1\Big((p-\varepsilon,\,p+\varepsilon)\cap\{u^*=0\}\Big)\geqslant\varepsilon. \tag{3.6}$$

For small $\varepsilon>0$, we can also suppose that $(p-\varepsilon,\,p+\varepsilon)\subset(0,1)$.

Now we take $\varphi\in C_0^\infty([-1,1])$ with $\varphi>0$ in $(-1,1)$ and $|\dot\varphi|\leqslant 1$. For any $\varepsilon>0$, we define $\varphi_\varepsilon(x):=\varphi\left(\frac{x-p}{\varepsilon}\right)$ and we remark that

$$\int_{p-\varepsilon}^{p+\varepsilon}|\dot\varphi_\varepsilon|^2\leqslant\frac{2}{\varepsilon}. \tag{3.7}$$

Let also

$$u_\varepsilon(x):=u^*(x)+\varepsilon^4\varphi_\varepsilon(x).$$

Notice that $u_\varepsilon\geqslant u^*$, and

$$\begin{aligned}\mathcal{M}_2(u_\varepsilon)-\mathcal{M}_2(u^*)&=\mathcal{L}^1\big((0,1)\cap\{u_\varepsilon>0\}\big)-\mathcal{L}^1\big((0,1)\cap\{u^*>0\}\big)\\&=\mathcal{L}^1\big(\{x\in(p-\varepsilon,\,p+\varepsilon)\text{ s.t. }u^*(x)=0\}\big)\in[\varepsilon,\,2\varepsilon],\end{aligned} \tag{3.8}$$

thanks to (3.6). Notice also that, in view of (3.3) and (3.4),

$$\mathcal{M}_2(u^*)\in\left[\frac12,\,1\right)$$

and so, if $\varepsilon>0$ is sufficiently small, we deduce from (3.8) that also

$$\mathcal{M}_2(u_\varepsilon)\in\left[\frac12,\,1\right).$$

Therefore, by (3.1) and (3.8),

$$\Phi_0\big(\mathcal{M}_2(u_\varepsilon)\big)-\Phi_0\big(\mathcal{M}_2(u^*)\big)=-\frac{\mathcal{M}_2(u_\varepsilon)-\mathcal{M}_2(u^*)}{4}\leqslant-\frac{\varepsilon}{4}. \tag{3.9}$$

On the other hand, recalling (3.2) and (3.7),

$$\begin{aligned}\int_\Omega|\dot u_\varepsilon|^2-\int_\Omega|\dot u^*|^2&=\int_{p-\varepsilon}^{p+\varepsilon}\left(2\varepsilon^4\dot u^*\,\dot\varphi_\varepsilon+\varepsilon^8|\dot\varphi_\varepsilon|^2\right)\\&\leqslant 2\varepsilon^4\sqrt{\int_\Omega|\dot u^*|^2}\sqrt{\int_{p-\varepsilon}^{p+\varepsilon}|\dot\varphi_\varepsilon|^2}+\varepsilon^8\int_{p-\varepsilon}^{p+\varepsilon}|\dot\varphi_\varepsilon|^2\leqslant 2\varepsilon^4\sqrt2\sqrt{\frac{2}{\varepsilon}}+2\varepsilon^7\leqslant\varepsilon^3,\end{aligned}$$

as soon as $\varepsilon>0$ is small enough. Using this and (3.9), we obtain that

$$J[u_\varepsilon]-J[u^*]\leqslant-\frac{\varepsilon}{4}+\varepsilon^3<0$$

if $\varepsilon>0$ is small enough, which is a contradiction with the minimality of u^*. This shows that no minimizer exists in this case.

4. Irregular free boundaries

In this section, we would like to remark that if Φ_0 is not monotone, then there may exist minimizers whose free boundary is not regular, even in dimension 2 (therefore, the result in Theorem 1.7) cannot be generalized to nonlinear problems for which Φ_0 is not monotone).

To make an explicit example, we consider the case in which $n=2$, $\Omega:=B_1\subset\mathbb{R}^2$, $\lambda_1:=\lambda_2:=Q:=1$ and $\bar u(x):=x_1x_2$. We also define

$$\begin{aligned}c_1&:=\int_{\partial B_1}\bar u^+,\\c_2&:=\frac{c_1}{2}\left[\int_{B_1}|\nabla\bar u|^2+1\right]^{-1},\\c_3&:=2+\frac{1}{4c_2}\\\text{and}\qquad c_\star&:=\min\left\{\frac{\pi}{4},\,\frac{c_1}{2c_3}\right\}.\end{aligned}$$

We remark that $c_\star < \pi/2$. We consider a smooth function $\phi_\star : [0,+\infty) \to [0,+\infty)$ with $\phi_\star(0) = 0$, $\phi_\star(r) > 0$ for any $r > 0$, $\phi_\star(\pi/4) = 2$ and

$$1 = \phi_\star(\pi/2) = \min_{[c_\star,\,+\infty)} \phi_\star. \tag{4.1}$$

Let also $\Phi(r_1, r_2) := \phi_\star(r_2)$. In this way, we have that $\Phi_0(r) = \phi_\star(r)$ and we observe that all our structural assumptions on Φ_0 are satisfied in this case, except the monotonicity.

We will show that

$$J[\bar u] = \min_{u-\bar u \in W_0^{1,2}(B_1)} J[u], \tag{4.2}$$

hence $\bar u$ is a minimizer for J in B_1 with respect to its own boundary values. Interestingly, the set $\{\bar u > 0\}$ is in this case a singular cone, which shows that the monotonicity assumption on Φ_0 cannot be dropped if one wishes to prove that the free boundary of minimizers in the plane is smooth.

To prove (4.2), we argue as follows. Let u be such that $u - \bar u \in W_0^{1,2}(B_1)$ and set $v := \min\{u, 1\}$. Then, $v - \bar u \in W_0^{1,2}(B_1)$, therefore $v = \bar u$ along ∂B_1 and thus, if ν is the exterior normal of B_1,

$$\begin{aligned} c_1 = \int_{\partial B_1} \bar u^+ = \int_{\partial B_1} v^+ = \int_{\partial B_1} v^+ x\cdot\nu = \int_{B_1} \operatorname{div}(v^+ x) = \int_{B_1} (2v^+ + \nabla v^+ \cdot x) \\ \leqslant 2\mathcal{L}^2(B_1 \cap \{v > 0\}) + \int_{B_1} |\nabla v^+|. \end{aligned} \tag{4.3}$$

Now we observe that

$$\begin{aligned} \int_{B_1} |\nabla v^+| = \int_{B_1} 2 \cdot \frac{1}{2\sqrt{c_2}} \cdot \sqrt{c_2}\,|\nabla v^+| \leqslant \int_{B_1\cap\{v>0\}} \left(\frac{1}{4c_2} + c_2 |\nabla v|^2 \right) \\ \leqslant \frac{1}{4c_2}\,\mathcal{L}^2(B_1 \cap \{v > 0\}) + c_2 \int_{B_1} |\nabla u|^2 \leqslant (c_3 - 2)\,\mathcal{L}^2(B_1 \cap \{v > 0\}) + c_2\, J[u]. \end{aligned}$$

By plugging this information into (4.3) and recalling that $\{v > 0\} = \{u > 0\}$, we obtain

$$c_1 \leqslant c_3\, \mathcal{L}^2(B_1 \cap \{u > 0\}) + c_2\, J[u].$$

This implies that either

$$c_3\, \mathcal{L}^2(B_1 \cap \{u > 0\}) \geqslant \frac{c_1}{2} \tag{4.4}$$

or

$$c_2\, J[u] \geqslant \frac{c_1}{2}. \tag{4.5}$$

If (4.4) is satisfied, then

$$\mathcal{M}_2(u) = \mathcal{L}^2(B_1 \cap \{u > 0\}) \geqslant \frac{c_1}{2c_3} \geqslant c_\star.$$

Consequently, by (4.1) and using that $\pi = \mathcal{L}^2(B_1)$, we have

$$\Phi_0(\mathcal{M}_2(u)) \geqslant 1 = \Phi_0(\pi/2) = \Phi_0(\mathcal{M}_2(\bar u)).$$

Therefore, since $\bar u$ is harmonic, we conclude that $J[u] \geqslant J[\bar u]$. This proves (4.2) in this case, and we now consider the case in which (4.5) holds true.

In this setting, we have that

$$J[u] \geqslant \frac{c_1}{2c_2} = \int_{B_1} |\nabla \bar u|^2 + 1 = \int_{B_1} |\nabla \bar u|^2 + \Phi_0(\pi/2) = J[\bar u].$$

This proves (4.2), as desired.

5. BMO GRADIENT ESTIMATES AND LIPSCHITZ CONTINUITY OF THE MINIMIZERS

In this section, we will prove that minimizers have gradient which is locally in BMO and, as a consequence, we obtain an estimate for the integral averages of the minimizers. This method is structurally quite different from the classical techniques in [1, 2], which obtain Lipschitz estimates in the linear case without using BMO estimates on the gradient of the solution.

From now on we assume that (1.15) and (1.16) hold true.

We remark that condition (1.16) is satisfied in all nontrivial cases, and then it links ϖ to quantities only depending on Ω and u_0. More precisely, condition (1.16) is satisfied provided that $\bar{u}^+ = u^+\big|_{\partial\Omega}$ has some positive mass along the boundary (and when this does not happen, the positive phase of the minimizer is trivial). Indeed, we have the following observation:

Lemma 5.1. *Let u be a minimizer in Ω for the functional J in* (1.8)*. Assume that Ω has Lipschitz boundary and that $\bar{u}^+$ has some positive mass along $\partial\Omega$. Then* (1.16) *is satisfied, with*

$$\varpi := \frac{1}{C_\Omega \left(C_\Omega\, J[\bar{u}] + 2\|\bar{u}^+\|_{L^\infty(\partial\Omega)} \displaystyle\int_{\partial\Omega} \bar{u}^+ \right)} \left(\int_{\partial\Omega} \bar{u}^+ \right)^2,$$

where $C_\Omega > 0$ is the trace constant for the domain Ω for the embedding of $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$.

Proof. First of all, Lemma 2.2 gives that u is subharmonic, hence so is u^+. Therefore

$$\|u^+\|_{L^\infty(\Omega)} = \|u^+\|_{L^\infty(\partial\Omega)} = \|\bar{u}^+\|_{L^\infty(\partial\Omega)}.$$

Moreover, by the minimality of u,

$$\int_{\Omega^+(u)} |\nabla u|^2 \leqslant \int_\Omega |\nabla u|^2 \leqslant J[u] \leqslant J[\bar{u}].$$

Now, let $\eta > 0$, to be chosen appropriately. By the trace inequality (see e.g. Theorem 1(ii) on page 258 of [11]) and the observations above, we have that

$$\begin{aligned}
\int_{\partial\Omega} \bar{u}^+ &= \|u^+\|_{L^1(\partial\Omega)} \leqslant C_\Omega\, \|u^+\|_{W^{1,1}(\Omega)} \\
&= C_\Omega \int_{\Omega^+(u)} \big(|\nabla u| + u\big) \leqslant C_\Omega \int_{\Omega^+(u)} \left(\eta|\nabla u|^2 + \frac{1}{4\eta} + \|u^+\|_{L^\infty(\Omega)} \right) \\
&\leqslant C_\Omega \left[\eta\, J[\bar{u}] + \left(\frac{1}{4\eta} + \|\bar{u}^+\|_{L^\infty(\partial\Omega)} \right) \mathcal{L}^n\big(\Omega^+(u)\big) \right].
\end{aligned}$$

Hence, we choose

$$\eta := \frac{\displaystyle\int_{\partial\Omega} \bar{u}^+}{2C_\Omega\, J[\bar{u}]}$$

and we obtain that

$$C_\Omega \left(\frac{C_\Omega\, J[\bar{u}]}{2\displaystyle\int_{\partial\Omega} \bar{u}^+} + \|\bar{u}^+\|_{L^\infty(\partial\Omega)} \right) \mathcal{L}^n\big(\Omega^+(u)\big) \geqslant \frac{1}{2} \int_{\partial\Omega} \bar{u}^+,$$

which gives the desired result. ∎

For any $x_0 \in \Omega$ and for any $\rho > 0$, we use the notation

$$(\nabla u)_{x_0,\rho} := \fint_{B_\rho(x_0)} \nabla u(x)\, dx \tag{5.1}$$

and we establish the BMO estimate claimed in Theorem 1.1:

Proof of Theorem 1.1. Let $B_r(x_0) \subseteq D \Subset \Omega$. Without loss of generality, we assume that u does not vanish identically, hence $\mathcal{L}^n(\Omega^+(u)) > 0$. We consider here the function $v \in W^{1,2}(B_r(x_0))$ which solves

$$\begin{cases} \Delta v = 0 & \text{in } B_r(x_0), \\ v = u & \text{on } \partial B_r(x_0). \end{cases} \tag{5.2}$$

We also extend v to be equal to u in $\Omega \setminus B_r(x_0)$. Since $v \in \mathcal{A}$, we have that $J[u] \leqslant J[v]$ and therefore

$$\begin{aligned}\int_{B_r(x_0)} \big(|\nabla u|^2 - |\nabla v|^2\big) &\leqslant \Phi_0\big(\mathcal{M}_2(v)\big) - \Phi_0\big(\mathcal{M}_2(u)\big)\\ &= \Phi_0\left(\lambda_2 \int_{\Omega\setminus B_r(x_0)} Q\chi_{\{u>0\}} + \lambda_2 \int_{B_r(x_0)} Q\chi_{\{v>0\}}\right)\\ &\quad -\Phi_0\left(\lambda_2 \int_{\Omega\setminus B_r(x_0)} Q\chi_{\{u>0\}} + \lambda_2 \int_{B_r(x_0)} Q\chi_{\{u>0\}}\right).\end{aligned} \tag{5.3}$$

Now we claim that

$$\int_{B_r(x_0)} \big(|\nabla u|^2 - |\nabla v|^2\big) \leqslant C_\star\, r^n, \tag{5.4}$$

for some $C_\star > 0$ (independent of r). To prove this, we first assume that $r > 0$ is so small that

$$\int_{\Omega\setminus B_r(x_0)} \chi_{\{u>0\}} \geqslant \frac{\mathcal{L}^n(\Omega^+(u))}{2} =: c_\star. \tag{5.5}$$

We stress that $c_\star > 0$ depends on ϖ (but not on r). We also set

$$C_0 := \sup_{\xi\in[\lambda_2 c_\star, \lambda_2\lambda_\Omega)} \Phi_0'(\xi).$$

Let us fix $a \geqslant \lambda_2 c_\star$ and $b, c \in [0, +\infty)$, with $b \geqslant c$ and $a + b < \lambda_2\lambda_\Omega$, then observe that

$$\Phi_0(a+b) - \Phi_0(a+c) = \int_c^b \Phi_0'(a+\tau)\, d\tau \leqslant C_0\,(b-c) \leqslant C_0\, b.$$

On the other hand, if $a \geqslant \lambda_2 c_\star$ and $b, c \in [0, +\infty)$, with $b \leqslant c$ and $a + c < \lambda_2\lambda_\Omega$, then we have that

$$\Phi_0(a+b) - \Phi_0(a+c) \leqslant 0,$$

due to (1.7). Therefore, for any $a \geqslant \lambda_2 c_\star$ and $b, c \in [0, +\infty)$, with $a + b,\, a + c < \lambda_2\lambda_\Omega$, we get

$$\Phi_0(a+b) - \Phi_0(a+c) \leqslant C_0\, b.$$

Utilizing this inequality with

$$\begin{aligned} a &:= \lambda_2 \int_{\Omega\setminus B_r(x_0)} Q\chi_{\{u>0\}},\\ b &:= \lambda_2 \int_{B_r(x_0)} Q\chi_{\{v>0\}}\\ \text{and}\quad c &:= \lambda_2 \int_{B_r(x_0)} Q\chi_{\{u>0\}}\end{aligned} \tag{5.6}$$

yields

$$\begin{aligned}&\Phi_0\left(\lambda_2 \int_{\Omega\setminus B_r(x_0)} Q\chi_{\{u>0\}} + \lambda_2 \int_{B_r(x_0)} Q\chi_{\{v>0\}}\right) - \Phi_0\left(\lambda_2 \int_{\Omega\setminus B_r(x_0)} Q\chi_{\{u>0\}} + \lambda_2 \int_{B_r(x_0)} Q\chi_{\{u>0\}}\right)\\ &\quad\leqslant C_0\,\lambda_2 \int_{B_r(x_0)} Q\chi_{\{v>0\}} \leqslant C_\star\, r^n,\end{aligned}$$

for some $C_\star > 0$ (possibly depending on ϖ in (1.16)). Plugging this into (5.3) we see that (5.4) is satisfied if r is chosen so small to fulfill (5.5) (say $r \in [0, r_0]$).

Now we complete the proof of (5.4) when $r > r_0$. In this case, we use the notation in (5.6) and we claim that

$$\Phi_0(a+b) \leqslant \hat{C}\,(b+1), \tag{5.7}$$

for some $\hat{C} > 0$, possibly depending on r_0, Q, λ_2 and Ω. To this goal, we define

$$a_0 := \lambda_2 \int_{\Omega\setminus B_{r_0}(x_0)} Q \quad\text{and}\quad b_0 := a + b - a_0.$$

Notice that the condition $r > r_0$ implies that $a_0 \geqslant a$ and so $b_0 \leqslant b$. We distinguish two cases, either $b_0 \leqslant 0$ or $b_0 > 0$. If $b_0 \leqslant 0$, we use (1.7) and we obtain

$$\Phi_0(a+b) = \Phi_0(a_0+b_0) \leqslant \Phi_0(a_0).$$

This implies (5.7) in this case. If instead $b_0 > 0$, we have

$$\Phi_0(a+b) = \Phi_0(a_0+b_0) = \int_0^{b_0} \Phi_0'(a_0+\tau)\,d\tau+\Phi_0(a_0) \leqslant \sup_{\xi\in[a_0,\lambda_2\lambda_\Omega)} \Phi_0'(\xi)\,b_0+\Phi_0(a_0) \leqslant \sup_{\xi\in[a_0,\lambda_2\lambda_\Omega)} \Phi_0'(\xi)\,b+\Phi_0(a_0),$$

which completes the proof of (5.7).

Now we observe that $b \leqslant C' r^n$, for some $C' > 0$. This fact, together with (5.7) and the assumption that $r > r_0$, gives that $\Phi_0(a+b) \leqslant \tilde{C}\, r^n$. Using this and the estimate in (5.3), we find that

$$\int_{B_r(x_0)} \big(|\nabla u|^2 - |\nabla v|^2\big) \leqslant \Phi_0\left(\lambda_2 \int_{\Omega\setminus B_r(x_0)} Q\chi_{\{u>0\}} + \lambda_2 \int_{B_r(x_0)} Q\chi_{\{v>0\}}\right) = \Phi_0(a+b) \leqslant \tilde{C}\, r^n,$$

which establishes (5.4) also when $r > r_0$.

In addition, by (5.2),

$$\int_{B_r(x_0)} \big(|\nabla u|^2 - |\nabla v|^2\big) = \int_{B_r(x_0)} \big(|\nabla u|^2 - |\nabla v|^2 + 2\nabla v\cdot\nabla(v-u)\big) = \int_{B_r(x_0)} |\nabla u - \nabla v|^2.$$

This and (5.4) yield that

$$\int_{B_r(x_0)} |\nabla u - \nabla v|^2 \leqslant C_\star\, r^n. \tag{5.8}$$

Now we use some techniques developed in [9]. We recall the notation in (5.1) and we observe that, using Hölder inequality,

$$\begin{aligned} \int_{B_r(x_0)} |(\nabla v)_{x_0,r} - (\nabla u)_{x_0,r}|^2 &= \int_{B_r(x_0)} \left| \frac{1}{\mathcal{L}^n(B_r(x_0))} \left(\int_{B_r(x_0)} \nabla v - \nabla u \right) \right|^2 \\ &\leqslant \frac{1}{\mathcal{L}^n(B_r(x_0))} \left(\int_{B_r(x_0)} |\nabla v - \nabla u| \right)^2 \leqslant \int_{B_r(x_0)} |\nabla v - \nabla u|^2. \end{aligned} \tag{5.9}$$

Furthermore, we recall the following Campanato growth type estimate (see e.g. Theorem 5.1 in [8]), valid for any $0 < r < R$,

$$\int_{B_r(x_0)} |\nabla v - (\nabla v)_{x_0,r}|^2 \leqslant C\,\left(\frac{r}{R}\right)^{n+\alpha} \int_{B_R(x_0)} |\nabla v - (\nabla v)_{x_0,R}|^2, \tag{5.10}$$

for suitable $\alpha \in (0,1)$ and $C > 1$.

Now, using (5.9) and possibly allowing C to be a universal constant varying from line to line, we have

$$\begin{aligned} &\int_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0,r}|^2 \\ \leqslant\;& C\,\left[\int_{B_r(x_0)} |\nabla u - \nabla v|^2 + \int_{B_r(x_0)} |\nabla v - (\nabla v)_{x_0,r}|^2 + \int_{B_r(x_0)} |(\nabla v)_{x_0,r} - (\nabla u)_{x_0,r}|^2 \right] \\ \leqslant\;& C\,\left[\int_{B_r(x_0)} |\nabla u - \nabla v|^2 + \int_{B_r(x_0)} |\nabla v - (\nabla v)_{x_0,r}|^2 \right]. \end{aligned}$$

So, using (5.10),

$$\int_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0,r}|^2 \leqslant C\,\left[\int_{B_r(x_0)} |\nabla u - \nabla v|^2 + \left(\frac{r}{R}\right)^{n+\alpha} \int_{B_R(x_0)} |\nabla v - (\nabla v)_{x_0,R}|^2 \right]. \tag{5.11}$$

Now we remark that

$$\begin{aligned} &\int_{B_R(x_0)} |\nabla v - (\nabla v)_{x_0,R}|^2 \\ \leqslant\;& C\,\left[\int_{B_R(x_0)} |\nabla v - \nabla u|^2 + \int_{B_R(x_0)} |\nabla u - (\nabla u)_{x_0,R}|^2 + \int_{B_R(x_0)} |(\nabla u)_{x_0,R} - (\nabla v)_{x_0,R}|^2 \right] \\ \leqslant\;& C\,\left[\int_{B_R(x_0)} |\nabla v - \nabla u|^2 + \int_{B_R(x_0)} |\nabla u - (\nabla u)_{x_0,R}|^2 \right], \end{aligned}$$

where (5.9) has been used once again.

Let us plug this into (5.11), and recall that $r \leqslant R$. We exploit (5.8), and conclude that

$$\begin{aligned}
&\int_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0,r}|^2 \\
\leqslant\;& C\left[\int_{B_r(x_0)} |\nabla u - \nabla v|^2 + \left(\frac{r}{R}\right)^{n+\alpha}\int_{B_R(x_0)} |\nabla v - \nabla u|^2 + \left(\frac{r}{R}\right)^{n+\alpha}\int_{B_R(x_0)} |\nabla u - (\nabla u)_{x_0,R}|^2\right] \\
\leqslant\;& C\left[\int_{B_R(x_0)} |\nabla u - \nabla v|^2 + + \left(\frac{r}{R}\right)^{n+\alpha}\int_{B_R(x_0)} |\nabla u - (\nabla u)_{x_0,R}|^2\right] \\
\leqslant\;& CR^n + C\left(\frac{r}{R}\right)^{n+\alpha}\int_{B_R(x_0)} |\nabla u - (\nabla u)_{x_0,R}|^2.
\end{aligned}$$

Therefore, defining

$$\psi(r) := \sup_{t\leqslant r}\int_{B_t(x_0)} |\nabla u - (\nabla u)_{x_0,t}|^2,$$

we have that

$$\psi(r) \leqslant CR^n + C\left(\frac{r}{R}\right)^{n+\alpha}\psi(R).$$

Thus, by Lemma 2.1 in Chapter 3 of [12], we conclude that there exist $c > 0$ and $R_0 > 0$ such that

$$\psi(r) \leqslant Cr^n\left(\frac{\psi(R)}{R^n} + 1\right)$$

for all $r \leqslant R \leqslant R_0$, and hence

$$\int_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0,r}|^2 \leqslant Cr^n$$

for some tame constant $C > 0$.

Therefore, by Hölder inequality,

$$⨍_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0,r}| \leqslant \sqrt{⨍_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0,r}|^2} \leqslant C,$$

up to renaming constants, as desired. □

Exploiting Theorem 1.1, we can now prove Corollary 1.2, by arguing as follows:

Proof of Corollary 1.2. By Theorem 1.1, we have that $\nabla u \in L^q_{\mathrm{loc}}(\Omega)$, for any $1 < q < +\infty$. Hence u is continuous. The modulus of continuity σ follows as in [14] and [2]. Therefore, $\Omega^+(u)$ is open and thus, if $x_0 \in \Omega^+(u)$, there exists $r > 0$ such that $\overline{B_r(x_0)} \subset \Omega^+(u)$. Consequently,

$$m := \min_{\overline{B_r(x_0)}} u > 0.$$

Let now $\phi \in C_0^\infty(B_r(x_0))$, $\varepsilon \in \mathbb{R}$ and $u_\varepsilon := u + \varepsilon\phi$. If $|\varepsilon| < \frac{m}{1+\|\phi\|_{L^\infty(\mathbb{R}^n)}}$, we have that $u_\varepsilon > 0$ in $B_r(x_0)$ and so

$$\Omega^+(u_\varepsilon) = \Omega^+(u).$$

This implies that, for small ε,

$$0 \leqslant J[u_\varepsilon] - J[u] = \int_\Omega \left(|\nabla u_\varepsilon|^2 - |\nabla u|^2\right)$$

and therefore

$$\int_\Omega \nabla u \cdot \nabla \phi = 0,$$

which shows that u is harmonic in $B_r(x_0)$, as desired.

Now we prove (1.17). For this we observe that, by the continuity of u, it follows that

$$\lim_{r\to 0} ⨍_{\partial B_r(x_0)} u = 0 \quad \text{for any } x_0 \in \Gamma.$$

Therefore

$$
\begin{aligned}
\frac{1}{r^{n-1}}\int_{\partial B_r(x_0)} u &= \int_0^r \frac{d}{dt}\left(\frac{1}{t^{n-1}}\int_{\partial B_t(x_0)} u(x)\, d\mathcal{H}^{n-1}(x)\right) dt\\
&= \int_0^r \frac{d}{dt}\left(\int_{\partial B_1} u(x_0+t\omega)\, d\mathcal{H}^{n-1}(\omega)\right) dt\\
&= \int_0^r \left(\int_{\partial B_1} \nabla u(x_0+t\omega)\cdot\omega\, d\mathcal{H}^{n-1}(\omega)\right) dt\\
&= \int_0^r \left(\frac{1}{t^n}\int_{\partial B_t} \nabla u(x_0+\nu)\cdot\nu\, d\mathcal{H}^{n-1}(\nu)\right) dt\\
&= \int_0^r \left(\frac{1}{t^n}\frac{d}{dt}\left(\int_{B_t} \nabla u(x_0+\nu)\cdot\nu\, d\mathcal{H}^{n-1}(\nu)\right)\right) dt\\
&= \int_0^r \left(\frac{1}{t^{n-1}}\frac{d}{dt}\left(\int_{B_t(x_0)} \nabla u(x)\cdot\frac{x-x_0}{|x-x_0|}\, d\mathcal{H}^{n-1}(x)\right)\right) dt.
\end{aligned}
\tag{5.12}
$$

Now, we notice that

$$
\int_{B_\varepsilon(x_0)} (\nabla u)_{x_0,\varepsilon}\cdot\frac{x-x_0}{|x-x_0|}\, dx = 0 \tag{5.13}
$$

by odd symmetry, for any $\varepsilon>0$. In consequence of this, we have that

$$
\begin{aligned}
\left|\frac{1}{\varepsilon^{n-1}}\int_{B_\varepsilon(x_0)} \nabla u(x)\cdot\frac{x-x_0}{|x-x_0|}\, dx\right| &= \left|\frac{1}{\varepsilon^{n-1}}\int_{B_\varepsilon(x_0)} (\nabla u(x)-(\nabla u)_{x_0,\varepsilon})\cdot\frac{x-x_0}{|x-x_0|}\, dx\right|\\
&\leqslant \varepsilon\left(\frac{1}{\varepsilon^n}\int_{B_\varepsilon(x_0)} |\nabla u(x)-(\nabla u)_{x_0,\varepsilon}|\, dx\right)\longrightarrow 0,
\end{aligned}
$$

as $\varepsilon\to 0$, thanks to the BMO estimate in Theorem 1.1.

Thus, an integration by parts gives that

$$
\begin{aligned}
&\int_0^r \left(\frac{1}{t^{n-1}}\frac{d}{dt}\left(\int_{B_t(x_0)} \nabla u(x)\cdot\frac{x-x_0}{|x-x_0|}\, d\mathcal{H}^{n-1}(x)\right)\right) dt\\
=\;& \frac{1}{r^{n-1}}\int_{B_r(x_0)} \nabla u(x)\cdot\frac{x-x_0}{|x-x_0|}\, d\mathcal{H}^{n-1}(x) + (n-1)\int_0^r \frac{1}{t^n}\int_{B_t(x_0)} \nabla u(x)\cdot\frac{x-x_0}{|x-x_0|}\, d\mathcal{H}^{n-1}(x)\, dt.
\end{aligned}
$$

So, recalling (5.12) and using again (5.13), we obtain that

$$
\begin{aligned}
\frac{1}{r^{n-1}}\int_{\partial B_r(x_0)} u &= \frac{1}{r^{n-1}}\int_{B_r(x_0)} \big(\nabla u(x)-(\nabla u)_{x_0,r}\big)\cdot\frac{x-x_0}{|x-x_0|}\, d\mathcal{H}^{n-1}(x)\\
&\quad +(n-1)\int_0^r \frac{1}{t^n}\int_{B_t(x_0)} \big(\nabla u(x)-(\nabla u)_{x_0,t}\big)\cdot\frac{x-x_0}{|x-x_0|}\, d\mathcal{H}^{n-1}(x)\, dt.
\end{aligned}
$$

Therefore, the BMO estimate in Theorem 1.1 yields the desired result in (1.17). ☐

As customary, one can deduce from the integral estimate in (1.17) a linear growth from the free boundary. We give the details for convenience, starting from the one-phase case:

Corollary 5.2. *Let $u\geqslant 0$ be a minimizer in Ω for the functional J in* (1.8) *and let $D\Subset\Omega$. Let $\varpi>0$ and assume that $\mathcal{L}^n\big(\Omega^+(u)\big)\geqslant\varpi$.*

Then, there exists $C>0$, possibly depending on ϖ, Q, Ω and D, such that

$$
u(x)\leqslant C\,\mathrm{dist}(x,\Gamma),
$$

for any $x\in D$ for which $B_{2\mathrm{dist}(x,\Gamma)}(x)\subset D$.

Proof. Let d be the distance of x to Γ. Let $x_0\in\overline{B_d(x)}\cap\Gamma$. Then, we can use (1.17) and obtain that, for any $\rho\in(0,2d)$

$$
\int_{\partial B_\rho(x_0)} u\leqslant C\rho^n,
$$

for some $C > 0$. So, we integrate this inequality in $\rho \in (0, 2d)$ and we find that

$$\int_{B_{2d}(x_0)} u \leqslant C d^{n+1}, \tag{5.14}$$

up to renaming $C > 0$.

On the other hand, since u is harmonic in $B_d(x)$, thanks to Corollary 1.2, we have that

$$u(x) = \fint_{B_d(x)} u. \tag{5.15}$$

Notice now that $B_d(x) \subseteq B_{2d}(x_0)$, hence we deduce from (5.14) and (5.15) that $u(x) \leqslant Cd$, as desired. □

From Corollary 5.2 one can deduce that that u is Lipschitz continuous, as stated in the next result for completeness:

Corollary 5.3. *Let $u \geqslant 0$ be a minimizer in Ω for the functional J in* (1.8) *and let $D \Subset \Omega$. Let $\varpi > 0$ and assume that $\mathcal{L}^n\big(\Omega^+(u)\big) \geqslant \varpi$.*

Then, there exists $C > 0$, possibly depending on ϖ, Q, Ω and D, such that

$$\sup_{x \in D} |\nabla u(x)| \leqslant C.$$

The proof of Corollary 5.3 is by now standard (see e.g. Theorem 5.3 in [2]).

The Lipschitz estimate in Corollary 5.3 is optimal, since the solutions have linear growth from the free boundary, as stated in the following result:

Lemma 5.4. *Let u be a minimizer in Ω for the functional J in* (1.8) *and let $D \Subset \Omega$.*

Let $d > 0$ and suppose that $B_d \subseteq \Omega^+(u)$.

Let $\bar\omega \geqslant \varpi > 0$ and assume that

$$\mathcal{L}^n\big(\Omega^+(u)\big) \in [\varpi, \bar\omega]. \tag{5.16}$$

Assume also that

$$\Upsilon := \inf_{r \in [\lambda_2 Q_1 \varpi/2,\ 2\lambda_2 Q_2 \bar\omega]} \Phi_0'(r) > 0. \tag{5.17}$$

Then, there exist d_0, $c_0 > 0$, possibly depending on ϖ, $\bar\omega$, Q, λ_1, λ_2 and Ω, such that if $d \in (0, d_0)$ we have that

$$u(0) \geqslant c_0\, \Upsilon\, d.$$

Proof. We let

$$C_0 := \lambda_2 \int_{\Omega \setminus B_{d/2}} Q \chi_{\{u>0\}}.$$

We take $d_0 > 0$ small enough such that

$$C_0 \geqslant \lambda_2\, Q_1\, \frac{\varpi}{2}.$$

In addition, we have that

$$C_0 \leqslant \lambda_2\, Q_2\, \bar\omega.$$

Then, for any $a, b \in [0,\ \lambda_2 Q_2 \bar\omega]$, with $a \geqslant b$, we have that

$$\Phi_0(C_0 + a) - \Phi_0(C_0 + b) \geqslant \Upsilon\, (a - b), \tag{5.18}$$

thanks to (5.17).

Also, from Corollary 1.2, we know that u is harmonic in B_d and so, by Harnack inequality,

$$\sup_{B_{d/2}} u \leqslant \bar C \inf_{B_{d/2}} u \leqslant \bar C u(0), \tag{5.19}$$

for some $\bar C > 0$.

We take $\psi_0 \in C^\infty(\mathbb{R}^n)$, such that $\psi_0 = 0$ in $B_{1/4}$, $\psi_0 = 1$ on $\partial B_{1/2}$ and $|\nabla \psi_0| \leqslant 10$. We also set

$$\psi(x) := 2\bar{C}\, u(0)\, \psi_0 \left(\frac{x}{d}\right).$$

Notice that

$$|\nabla \psi| \leqslant \frac{20\bar{C}\, u(0)}{d}. \tag{5.20}$$

If $x \in B_{d/2}$, we define

$$v(x) := \min\{u(x),\, \psi(x)\}.$$

Notice that if $x \in \partial B_{d/2}$, then

$$\psi(x) = 2\bar{C}\, u(0) \geqslant u(x),$$

thanks to (5.19), hence $v = u$ on $\partial B_{d/2}$. Therefore, we extend $v(x) := u(x)$ for any x outside $B_{d/2}$, and we have that

$$\begin{aligned} 0 \leqslant J[v] - J[u] &= \int_\Omega |\nabla v|^2 - \int_\Omega |\nabla u|^2 + \Phi_0(\mathcal{M}_2(v)) - \Phi_0(\mathcal{M}_2(u)) \\ &= \int_{B_{d/2}\cap\{\psi>u\}} |\nabla \psi|^2 - \int_{B_{d/2}\cap\{\psi>u\}} |\nabla u|^2 + \Phi_0(\mathcal{M}_2(v)) - \Phi_0(\mathcal{M}_2(u)). \end{aligned} \tag{5.21}$$

Now, from (5.20), we have that

$$\int_{B_{d/2}\cap\{\psi>u\}} |\nabla \psi|^2 - \int_{B_{d/2}\cap\{\psi>u\}} |\nabla u|^2 \leqslant \int_{B_{d/2}} |\nabla \psi|^2 \leqslant \frac{400\bar{C}^2\, \mathcal{L}^n(B_{d/2})\, u^2(0)}{d^2}. \tag{5.22}$$

On the other hand

$$\begin{aligned} \Phi_0(\mathcal{M}_2(u)) - \Phi_0(\mathcal{M}_2(v)) &= \Phi_0\left(\lambda_2 \int_\Omega Q\chi_{\{u>0\}}\right) - \Phi_0\left(\lambda_2 \int_\Omega Q\chi_{\{v>0\}}\right) \\ &= \Phi_0\left(C_0 + \lambda_2 \int_{B_{d/2}} Q\chi_{\{u>0\}}\right) - \Phi_0\left(C_0 + \lambda_2 \int_{B_{d/2}} Q\chi_{\{v>0\}}\right). \end{aligned}$$

Notice that the quantity

$$\lambda_2 \int_{B_{d/2}} Q\chi_{\{u>0\}} + \lambda_2 \int_{B_{d/2}} Q\chi_{\{v>0\}}$$

is small if d is small enough, and so we can apply (5.18) with $a := C_0 + \lambda_2 \int_{B_{d/2}} Q\chi_{\{u>0\}}$ and $b := C_0 + \lambda_2 \int_{B_{d/2}} Q\chi_{\{v>0\}}$. In this way, we find that

$$\begin{aligned} \Phi_0(\mathcal{M}_2(u)) - \Phi_0(\mathcal{M}_2(v)) &\geqslant \Upsilon\lambda_2 \int_{B_{d/2}} Q\left(\chi_{\{u>0\}} - \chi_{\{v>0\}}\right) \\ &= \lambda_2 \int_{B_{d/2}\cap\{u>\psi\}} Q\left(\chi_{\{u>0\}} - \chi_{\{\psi>0\}}\right). \end{aligned}$$

Notice also that in $B_{d/4}$ we have that $\psi = 0 < u$, hence we conclude that

$$\Phi_0(\mathcal{M}_2(u)) - \Phi_0(\mathcal{M}_2(v)) \geqslant \lambda_2 \int_{B_{d/4}} Q\left(\chi_{\{u>0\}} - \chi_{\{\psi>0\}}\right) \geqslant \lambda_2\, Q_1\, \mathcal{L}^n(B_{d/4}).$$

Now, plugging this and (5.22) into (5.21) we infer

$$\frac{400\bar{C}^2\, \mathcal{L}^n(B_{d/2})\, u^2(0)}{d^2} \geqslant \lambda_2\, Q_1\, \mathcal{L}^n(B_{d/4}),$$

which implies the desired result. □

The Lipschitz regularity for the pure two-phase problem, as stated in Theorem 1.3, can be deduced from the BMO estimate, giving coherent growth for u^+ and u^-, and the classical Alt-Caffarelli-Friedman monotonicity formula. The details go as follows:

Proof of Theorem 1.3. First we observe that, from (1.17), it follows that

$$\left| \frac{1}{r^{n-1}} \int_{\partial B_r(x)} u^+ - \frac{1}{r^{n-1}} \int_{\partial B_r(x)} u^- \right| \leqslant C, \tag{5.23}$$

for any $x \in \Gamma$ and $r > 0$ such that $B_r(x) \subseteq D \Subset \Omega$.

We recall now the Alt-Caffarelli-Friedman monotonicity formula [2]: Let w^+, w^- be two continuous, nonnegative subharmonic functions in B_1, with $w_1 w_2 = 0$, $w_1(0) = w_2(0) = 0$. Then, for any $x_0 \in \Gamma$,

$$\Phi(r, w_1, w_2) := \frac{1}{r^4} \int_{B_r(x_0)} \frac{|\nabla w_1(x)|^2}{|x - x_0|^{n-2}}\, dx \int_{B_r(x_0)} \frac{|\nabla w_2(x)|^2}{|x - x_0|^{n-2}}\, dx$$

is a monotone increasing function of $r \in (0, 1)$, and

(5.24)
$$\Phi(1, w_1, w_2) \leqslant C \left(1 + \int_{B_1} w_1^2 + \int_{B_1} w_2^2\right),$$

with some universal constant $C > 0$.

In what follows, we will apply this theorem with $w_1 := u^+, w_2 := u^-$.

To fix the ideas we assume that $B_1 \subset D$. Moreover, we take $x_0 \in D$ such that $u(x_0) > 0$ and let $x \in \Gamma = \partial\{u > 0\}$ be the closest point to x_0, that is dist$(x_0, \Gamma) = |x_0 - x|$.

Setting $\rho := |x - x_0|$, we suppose that $u(x_0) \geqslant M\rho > 0$, for some large $M > 0$. Hence, applying the Harnack's inequality, we infer that

(5.25)
$$u \geqslant c_0 M \rho \quad \text{in } B_{\frac{3\rho}{4}}(x_0) \subset D,$$

for some $c_0 > 0$. Therefore, setting also

$$\Sigma_\rho := \partial B_\rho(x) \cap B_{\frac{3\rho}{4}}(x_0),$$

we conclude that

$$\fint_{\partial B_\rho(x)} u^+ \geqslant c_1 \fint_{\Sigma_\rho} u^+ \geqslant c_0 c_1 M \rho,$$

where $c_1 > 0$ depends only on the dimension n.

From this and (5.23), we obtain that

(5.26)
$$\fint_{\partial B_\rho(x)} u^- \geqslant \fint_{\partial B_\rho(x)} u^+ - C\rho \geqslant (c_0 c_1 M - C)\rho > \frac{M}{2}\rho$$

if M is large enough.

Let $y \in \partial B_{\frac{\rho}{2}}(x_0) \cap [x, x_0]$ be the mid-point of the segment $[x, x_0]$. Then, by construction, we have that

(5.27)
$$u^+ \geqslant c_0 M \rho \quad \text{in } B_{\frac{\rho}{4}}(y),$$

where c_0 is the constant in (5.25).

For our next computation, it is convenient to switch to polar coordinates (r, σ) centered at x. Let E_ρ be the set of $\sigma \in S^{n-1}$ such that $u(x + \rho\sigma) < 0$. Let also I_σ be the ray that connects y and $x + \rho\sigma$. In what follows, we parameterize I_σ in arc-length by the parameter $r \geqslant 0$, with $r = 0$ corresponding to the point y. The function u evaluated at the point of I_σ parameterized by r will be denoted by $u(r)$.

In this notation, formula (5.27) says that $u^-(r) = 0$ for any $r \in \left(0, \frac{\rho}{4}\right)$, and so

$$\nabla u^-(r) = 0 \quad \text{for any } r \in \left(0, \frac{\rho}{4}\right).$$

Then, recalling (5.26), we have that

$$
\begin{aligned}
\frac{M}{2} &\leqslant \frac{1}{\rho} \fint_{\partial B_\rho(x)} u^- = \frac{C}{\rho} \int_{E_\rho} u^-(x+\rho\sigma)\, d\mathcal{H}^{n-1}(\sigma) = \frac{C}{\rho} \int_{E_\rho} \int_{I_\sigma \cap B_\rho(x)} D_r u^-(r)\, dr\, d\mathcal{H}^{n-1}(\sigma) \\
&\leqslant \frac{C}{\rho} \left(\rho \mathcal{H}^{n-1}(E_\rho)\right)^{\frac{1}{2}} \left(\int_{E_\rho} \int_{I_\sigma \cap B_\rho(x)} |D_r u^-(r)|^2\, dr\, d\mathcal{H}^{n-1}(\sigma) \right)^{\frac{1}{2}} \\
&\leqslant \frac{C}{\rho} \left(\rho \mathcal{H}^{n-1}(E_\rho)\right)^{\frac{1}{2}} \left(\int_{S^{n-1}} \int_{I_\sigma \cap \{r\in[\rho/4,\, 2\rho]\}} |D_r u^-(r)|^2\, dr\, d\mathcal{H}^{n-1}(\sigma) \right)^{\frac{1}{2}} \\
&\leqslant \frac{C}{\rho} \left(\rho \mathcal{H}^{n-1}(E_\rho)\right)^{\frac{1}{2}} \left(\int_{B_{2\rho}(y)\setminus B_{\rho/4}(y)} \frac{|\nabla u^-(z)|^2}{|z-y|^{n-1}}\, dz \right)^{\frac{1}{2}} \\
&\leqslant \frac{C}{\rho} \left(\rho \mathcal{H}^{n-1}(E_\rho)\right)^{\frac{1}{2}} \left(\frac{1}{\rho} \int_{B_{2\rho}(y)\setminus B_{\rho/4}(y)} \frac{|\nabla u^-(z)|^2}{|z-y|^{n-2}}\, dz \right)^{\frac{1}{2}},
\end{aligned}
\tag{5.28}
$$

up to renaming $C > 0$ from line to line, where the Hölder's inequality was also used.

Now we observe that, if $z \in B_{2\rho}(y) \setminus B_{\rho/4}(y)$, we have that

$$
|z-x| \leqslant |z-y| + |x-y| \leqslant 3\rho
$$

and so

$$
|z-y| \geqslant \frac{\rho}{4} \geqslant \frac{|z-x|}{12}.
$$

Thus, renaming constants in (5.28), we obtain that

$$
\begin{aligned}
\frac{M}{2} &\leqslant \frac{C}{\rho} \left(\rho \mathcal{H}^{n-1}(E_\rho)\right)^{\frac{1}{2}} \left(\frac{1}{\rho} \int_{B_{3\rho}(x)} \frac{|\nabla u^-(z)|^2}{|z-x|^{n-2}}\, dz \right)^{\frac{1}{2}} \\
&= \frac{C}{\rho} \left(\mathcal{H}^{n-1}(E_\rho)\right)^{\frac{1}{2}} \left(\int_{B_{3\rho}(x)} \frac{|\nabla u^-(z)|^2}{|z-x|^{n-2}}\, dz \right)^{\frac{1}{2}}.
\end{aligned}
\tag{5.29}
$$

In order to estimate the integral average of u^+, we use (5.27) and we obtain that (up to renaming constants)

$$
\begin{aligned}
M\rho^n &\leqslant C \int_{\partial B_{\frac{\rho}{4}}(y)} u^+ = C\rho^{n-1} \int_{\partial B_1} u^+ \left(y + \frac{\rho}{4}\omega\right) d\mathcal{H}^{n-1}(\omega) \\
&= C\rho^{n-1} \int_{\partial B_1} \left[u^+ \left(y + \frac{\rho}{4}\omega\right) - u^+(x) \right] d\mathcal{H}^{n-1}(\omega) \\
&\leqslant C\rho^{n-1} \int_{\partial B_1} \left[\int_0^1 \left| \nabla u^+ \left(x + r \left(y - x + \frac{\rho}{4}\omega \right) \right) \cdot \left(y - x + \frac{\rho}{4}\omega \right) \right| dr \right] d\mathcal{H}^{n-1}(\omega) \\
&\leqslant C\rho^{n} \int_{\partial B_1} \left[\int_0^1 \left| \nabla u^+ \left(x + r \left(y - x + \frac{\rho}{4}\omega \right) \right) \right| dr \right] d\mathcal{H}^{n-1}(\omega) \\
&\leqslant C\rho^{n-1} \int_{\partial B_1} \left[\int_{\rho/4}^{2\rho} \left| \nabla u^+ \left(x + \bar r \bar\omega \right) \right| d\bar r \right] d\mathcal{H}^{n-1}(\bar\omega) \\
&= C\rho^{n-1} \int_{\partial B_1} \left[\int_{\rho/4}^{2\rho} \frac{\bar r^{n-1}\ \left| \nabla u^+ \left(x + \bar r \bar\omega \right) \right|}{\bar r^{n-1}} d\bar r \right] d\mathcal{H}^{n-1}(\bar\omega) \\
&= C\rho^{n-1} \int_{B_{2\rho}(x)\setminus B_{\rho/4}(x)} \frac{\left| \nabla u^+ (z) \right|}{|z-x|^{n-1}}\, dz \\
&\leqslant C\rho^{n-1+\frac{n}{2}} \left(\int_{B_{2\rho}(x)\setminus B_{\rho/4}(x)} \frac{\left| \nabla u^+ (z) \right|^2}{|z-x|^{2(n-1)}}\, dz \right)^{\frac{1}{2}} \\
&\leqslant C\rho^{n-1} \left(\int_{B_{2\rho}(x)\setminus B_{\rho/4}(x)} \frac{\left| \nabla u^+ (z) \right|^2}{|z-x|^{n-2}}\, dz \right)^{\frac{1}{2}}.
\end{aligned}
$$

Combining this with (5.29), and renaming constants, we conclude that

$$\begin{aligned} M^2 &\leqslant \frac{C}{\rho^2}\left(\mathcal{H}^{n-1}(E_\rho)\right)^{\frac12}\left(\int_{B_{3\rho}(x)}\frac{|\nabla u^-(z)|^2}{|z-x|^{n-2}}\,dz\right)^{\frac12}\left(\int_{B_{2\rho}(x)\setminus B_{\rho/4}(x)}\frac{\left|\nabla u^+(z)\right|^2}{|z-x|^{n-2}}\,dz\right)^{\frac12}\\ &\leqslant \frac{C}{\rho}\left(\int_{B_{3\rho}(x)}\frac{|\nabla u^-(z)|^2}{|z-x|^{n-2}}\,dz\right)^{\frac12}\left(\int_{B_{3\rho}(x)}\frac{\left|\nabla u^+(z)\right|^2}{|z-x|^{n-2}}\,dz\right)^{\frac12}\\ &= C\left(\Phi(\rho,u^+,u^-)\right)^{\frac12}. \end{aligned}$$

Hence, from the monotonicity formula and (5.24), we get that

$$M^2 \leqslant C\left(1+\int_{B_1(x)}(u^+)^2+\int_{B_1(x)}(u^-)^2\right)^{\frac12},$$

which bounds M, as desired. □

6. Free boundary condition

In this section, we will assume that the function Q introduced in (1.4) is continuous.

Next result shows that, at points p of the free boundary, the following condition holds true in the sense of distributions:

$$\begin{aligned} &\left(\partial^+_\nu u(p)\right)^2-\left(\partial^-_\nu u(p)\right)^2\\ &=\left[\lambda_2\partial_{r_2}\Phi\left(\lambda_1\int_\Omega Q\,\chi_{\{u<0\}},\ \lambda_2\int_\Omega Q\,\chi_{\{u>0\}}\right)\ -\ \lambda_1\partial_{r_1}\Phi\left(\lambda_1\int_\Omega Q\,\chi_{\{u<0\}},\ \lambda_2\int_\Omega Q\,\chi_{\{u>0\}}\right)\right]Q(p), \end{aligned}$$

where ν is the normal vector exterior to $\partial\{u>0\}$ (and thus pointing towards $\{u\leqslant 0\}$) and we set

$$\partial^+_\nu u(x):=\lim_{t\to0}\frac{u(x-t\nu)-u(x)}{t}\quad\text{and}\quad \partial^-_\nu u(x):=\lim_{t\to0}\frac{u(x+t\nu)-u(x)}{t}. \tag{6.1}$$

More precisely, we have that:

Lemma 6.1. *Let u be a minimizer of J as in* (1.8) *and suppose that $Q\in C^1(\Omega)$. Assume also that*

$$\text{the set } \{u=0\}\cap\Omega \text{ has zero measure.} \tag{6.2}$$

Then

$$\lim_{\varepsilon\searrow0}\left\{\int_{\partial\{u>\varepsilon\}\cap\Omega}\mathcal{I}^{\varepsilon,+}(u,x)\,V(x)\cdot\nu^{\varepsilon,+}(x)\,d\mathcal{H}^{n-1}(x)+\int_{\partial\{u<-\varepsilon\}\cap\Omega}\mathcal{I}^{\varepsilon,-}(u,x)\,V(x)\cdot\nu^{\varepsilon,-}(x)\,d\mathcal{H}^{n-1}(x)\right\}=0$$

for any vector field $V\in C^\infty_0(\Omega,\mathbb{R}^n)$, where we have denoted by $\nu^{\varepsilon,+}$ and $\nu^{\varepsilon,-}$ the exterior normals of $\{u>\varepsilon\}$ and $\{u<-\varepsilon\}$, respectively, and

$$\begin{aligned} &\mathcal{I}^{\varepsilon,+}(u,x):=|\partial^+_\nu u(x)|^2-\lambda_2\partial_{r_2}\Phi\left(\lambda_1\int_\Omega Q(\xi)\,\chi_{\{u<-\varepsilon\}}(\xi)\,d\xi,\ \lambda_2\int_\Omega Q(\xi)\,\chi_{\{u>\varepsilon\}}(\xi)\,d\xi\right)Q(x)\\ \text{and }\ &\mathcal{I}^{\varepsilon,-}(u,x):=|\partial^-_\nu u(x)|^2-\lambda_1\partial_{r_1}\Phi\left(\lambda_1\int_\Omega Q(\xi)\,\chi_{\{u<-\varepsilon\}}(\xi)\,d\xi,\ \lambda_2\int_\Omega Q(\xi)\,\chi_{\{u>\varepsilon\}}(\xi)\,d\xi\right)Q(x). \end{aligned} \tag{6.3}$$

Proof. The argument is a (not completely straightforward) modification of the classical domain variations in Theorem 2.5 in [1] and in Theorem 2.4 of [2]. We provide full details for the facility of the reader. For small $t\in\mathbb{R}$, we consider the ODE flow $y=y(t;x)$ given by the Cauchy problem

$$\begin{cases}\partial_t y(t;x)=V(y(t;x)),\\ \qquad y(0;x)=x.\end{cases}$$

The map $\mathbb{R}^n\ni x\mapsto y(t;x)$ is invertible for small t, i.e. we can consider the inverse diffeomorphism $x(t;y)$ and we define

$$u_t(y):=u(x(t;y)).$$

We remark that, in light of (6.2),

$$\text{the set } \{u_t=0\}\cap\Omega \text{ has zero measure.} \tag{6.4}$$

Given $\varepsilon > 0$, we define $E^{\varepsilon,+} := \{u > \varepsilon\} \cap \Omega$, $E^{\varepsilon,-} := \{u < \varepsilon\} \cap \Omega$ and $E_t^{\varepsilon,\pm} := y(t; E^{\varepsilon,\pm})$. Notice that

$$\{u_t > \varepsilon\} \cap \Omega = E_t^{\varepsilon,+} \text{ and } \{u_t < \varepsilon\} \cap \Omega = E_t^{\varepsilon,-}. \tag{6.5}$$

One can check (see e.g. formulas (4.5), (4.13) and (4.22) in [10]) that

$$y(t;\Omega) = \Omega, \tag{6.6}$$

$$\det D_x y(t;x) = 1 + t\,\mathrm{div}V(x) + o(t) \tag{6.7}$$

$$\text{and} \quad \int_\Omega |\nabla u(x)|^2\,dx - \int_\Omega |\nabla u_t(y)|^2\,dy = \lim_{\varepsilon\searrow 0} t \int_{\partial E^{\varepsilon,+}\cap\Omega} |\partial_\nu^+ u(y)|^2\, V(y)\cdot \nu^{\varepsilon,+}(y)\,d\mathcal{H}^{n-1}(y) \tag{6.8}$$
$$+t \int_{\partial E^{\varepsilon,-}\cap\Omega} |\partial_\nu^- u(y)|^2\, V(y)\cdot \nu^{\varepsilon,-}(y)\,d\mathcal{H}^{n-1}(y) + o(t).$$

By (6.6) and (6.7), we have that

$$\begin{aligned}
\int_\Omega Q(y)\,\chi_{E_t^{\varepsilon,\pm}}(y)\,dy &= \int_\Omega Q\big(y(t;x)\big)\,\chi_{E^{\varepsilon,\pm}}(x)\,\big(1+t\,\mathrm{div}V(x)+o(t)\big)\,dx\\
&= \int_\Omega \big(Q(x)+t\nabla Q(x)\cdot V(x)+o(t)\big)\,\chi_{E^{\varepsilon,\pm}}(x)\,\big(1+t\,\mathrm{div}V(x)+o(t)\big)\,dx\\
&= \int_\Omega \big(Q(x)+t\nabla Q(x)\cdot V(x)+tQ(x)\,\mathrm{div}V(x)+o(t)\big)\,\chi_{E^{\varepsilon,\pm}}(x)\,dx\\
&= \int_\Omega \big(Q(x)+t\,\mathrm{div}\big(Q(x)V(x)\big)+o(t)\big)\,\chi_{E^{\varepsilon,\pm}}(x)\,dx.
\end{aligned}$$

Consequently, we can linearize Φ and obtain

(6.9)
$$\begin{aligned}
&\Phi\left(\lambda_1\int_\Omega Q(y)\,\chi_{E_t^{\varepsilon,-}}(y)\,dy,\ \lambda_2\int_\Omega Q(y)\,\chi_{E_t^{\varepsilon,+}}(y)\,dy\right)\\
&= \Phi\left(\lambda_1\int_\Omega Q(x)\,\chi_{E^{\varepsilon,-}}(x)\,dx,\ \lambda_2\int_\Omega Q(x)\,\chi_{E^{\varepsilon,+}}(x)\,dx\right)\\
&\quad + t\lambda_1\partial_{r_1}\Phi\left(\lambda_1\int_\Omega Q(x)\,\chi_{E^{\varepsilon,-}}(x)\,dx,\ \lambda_2\int_\Omega Q(x)\,\chi_{E^{\varepsilon,+}}(x)\,dx\right)\int_\Omega \mathrm{div}\big(Q(x)V(x)\big)\,\chi_{E^{\varepsilon,-}}(x)\,dx\\
&\quad + t\lambda_2\partial_{r_2}\Phi\left(\lambda_1\int_\Omega Q(x)\,\chi_{E^{\varepsilon,-}}(x)\,dx,\ \lambda_2\int_\Omega Q(x)\,\chi_{E^{\varepsilon,+}}(x)\,dx\right)\int_\Omega \mathrm{div}\big(Q(x)V(x)\big)\,\chi_{E^{\varepsilon,+}}(x)\,dx + o(t)\\
&= \Phi\left(\lambda_1\int_\Omega Q(x)\,\chi_{E^{\varepsilon,-}}(x)\,dx,\ \lambda_2\int_\Omega Q(x)\,\chi_{E^{\varepsilon,+}}(x)\,dx\right)\\
&\quad + t\lambda_1\partial_{r_1}\Phi\left(\lambda_1\int_\Omega Q(x)\,\chi_{E^{\varepsilon,-}}(x)\,dx,\ \lambda_2\int_\Omega Q(x)\,\chi_{E^{\varepsilon,+}}(x)\,dx\right)\int_{\partial E^{\varepsilon,-}\cap\Omega} Q(x)V(x)\cdot\nu^{\varepsilon,-}(x)\,d\mathcal{H}^{n-1}(x)\\
&\quad + t\lambda_2\partial_{r_2}\Phi\left(\lambda_1\int_\Omega Q(x)\,\chi_{E^{\varepsilon,-}}(x)\,dx,\ \lambda_2\int_\Omega Q(x)\,\chi_{E^{\varepsilon,+}}(x)\,dx\right)\int_{\partial E^{\varepsilon,+}\cap\Omega} Q(x)V(x)\cdot\nu^{\varepsilon,+}(x)\,d\mathcal{H}^{n-1}(x)\\
&\qquad + o(t).
\end{aligned}$$

Moreover, by inspection and recalling (6.5), one sees that

$$\lim_{\varepsilon\searrow 0}\chi_{E_t^{\varepsilon,+}} = \lim_{\varepsilon\searrow 0}\chi_{\{u_t>\varepsilon\}} = \chi_{\{u_t>0\}},$$

for any small $t \geqslant 0$. Similarly, and using (6.4),

$$\lim_{\varepsilon\searrow 0}\chi_{E_t^{\varepsilon,-}} = \chi_{\{u_t<0\}} = \chi_{\{u_t\leqslant 0\}}$$

a.e. in Ω. As a consequence, by the Dominated Convergence Theorem,

$$\mathcal{M}_1(u_t) = \lim_{\varepsilon\searrow 0}\lambda_1\int_\Omega Q(x)\,\chi_{E_t^{\varepsilon,-}}(x)\,dx \quad\text{and}\quad \mathcal{M}_2(u_t) = \lim_{\varepsilon\searrow 0}\lambda_2\int_\Omega Q(x)\,\chi_{E_t^{\varepsilon,+}}(x)\,dx,$$

for any small $t \geqslant 0$. So, we can take the limit with respect to ε in formula (6.9) and obtain that

$$\begin{aligned}
&\Phi\big(\mathcal{M}_1(u_t),\ \mathcal{M}_2(u_t)\big) - \Phi\big(\mathcal{M}_1(u),\ \mathcal{M}_2(u)\big)\\
=\ & \lim_{\varepsilon\searrow 0} t\lambda_1\partial_{r_1}\Phi\left(\lambda_1\int_\Omega Q(x)\,\chi_{E^{\varepsilon,-}}(x)\,dx,\ \lambda_2\int_\Omega Q(x)\,\chi_{E^{\varepsilon,+}}(x)\,dx\right)\int_{\partial E^{\varepsilon,-}\cap\Omega} Q(x)V(x)\cdot\nu^{\varepsilon,-}(x)\,d\mathcal{H}^{n-1}(x)\\
&\quad +t\lambda_2\partial_{r_2}\Phi\left(\lambda_1\int_\Omega Q(x)\,\chi_{E^{\varepsilon,-}}(x)\,dx,\ \lambda_2\int_\Omega Q(x)\,\chi_{E^{\varepsilon,+}}(x)\,dx\right)\int_{\partial E^{\varepsilon,+}\cap\Omega} Q(x)V(x)\cdot\nu^{\varepsilon,+}(x)\,d\mathcal{H}^{n-1}(x)\\
&\quad +o(t).
\end{aligned}$$

From this and (6.8), we have that

$$\begin{aligned}
&J[u]-J[u_t]\\
=\ & \int_\Omega |\nabla u(x)|^2\,dx - \int_\Omega |\nabla u_t(y)|^2\,dy\\
&\quad +\Phi\big(\mathcal{M}_1(u),\ \mathcal{M}_2(u)\big) - \Phi\big(\mathcal{M}_1(u_t),\ \mathcal{M}_2(u_t)\big)\\
=\ & \lim_{\varepsilon\searrow 0} t\int_{\partial E^{\varepsilon,+}\cap\Omega} |\partial_\nu^+ u(y)|^2\,V(y)\cdot\nu^{\varepsilon,+}(y)\,d\mathcal{H}^{n-1}(y) + t\int_{\partial E^{\varepsilon,-}\cap\Omega} |\partial_\nu^- u(y)|^2\,V(y)\cdot\nu^{\varepsilon,-}(y)\,d\mathcal{H}^{n-1}(y)\\
&\quad -t\lambda_1\partial_{r_1}\Phi\left(\lambda_1\int_\Omega Q(x)\,\chi_{E^{\varepsilon,-}}(x)\,dx,\ \lambda_2\int_\Omega Q(x)\,\chi_{E^{\varepsilon,+}}(x)\,dx\right)\int_{\partial E^{\varepsilon,-}\cap\Omega} Q(x)V(x)\cdot\nu^{\varepsilon,-}(x)\,d\mathcal{H}^{n-1}(x)\\
&\quad -t\lambda_2\partial_{r_2}\Phi\left(\lambda_1\int_\Omega Q(x)\,\chi_{E^{\varepsilon,-}}(x)\,dx,\ \lambda_2\int_\Omega Q(x)\,\chi_{E^{\varepsilon,+}}(x)\,dx\right)\int_{\partial E^{\varepsilon,+}\cap\Omega} Q(x)V(x)\cdot\nu^{\varepsilon,+}(x)\,d\mathcal{H}^{n-1}(x)\\
&\quad +o(t).
\end{aligned}$$

Dividing by t and then letting $t\to 0$, we obtain the desired result. □

We observe that when $\Phi(r_1,r_2) := r_1+r_2$, then (6.3) reduces to $\mathcal{I}^{\varepsilon,+}(u,x) := |\partial_\nu^+ u(x)|^2 - \lambda_2\,Q(x)$ and $\mathcal{I}^{\varepsilon,-}(u,x) := |\partial_\nu^- u(x)|^2 - \lambda_1\,Q(x)$. Hence, in this particular case, our Lemma 6.1 boils down to Theorem 2.4 in [2]. If also $\lambda_1 := 0$, then Lemma 6.1 boils down to Theorem 2.5 in [1].

Next we show that $\partial\{u>0\}$ contains $\partial\{u<0\}$ under the condition (1.7). Thus one has sharp separation of phases.

Lemma 6.2. *Let u be a minimizer in Ω of the functional in* (1.8)*. Then $\partial\{u<0\}\setminus\partial\{u>0\}=\varnothing$.*

Proof. Let $E := \partial\{u<0\}\setminus\partial\{u>0\}$. We want to show that E is empty and suppose by contradiction that $E\neq\varnothing$. Then, there exist $p\in\Omega$ and $r>0$ such that $u\leqslant 0$ in $B_r(p)$, with $u(p)=0$ and $\mathcal{L}^n\big(B_r(p)\cap\{u<0\}\big)>0$. Then, we use that u is subharmonic, in view of Lemma 2.2, and we obtain that

$$u(p)\leqslant \fint_{B_r(p)} u < 0,$$

which is a contradiction. □

7. Nondegeneracy of minimizers

One of the fundamental properties of the minimizers is a linear lower bound. In other words, the minimizers grow at least linearly away from the free boundary. This is the content of Theorem 1.4, that we now prove:

Proof of Theorem 1.4. Let $\kappa\in(0,1)$. Without loss of generality we assume that $x_0=0$.

Notice that the critical points of a non-constant harmonic function have Hausdorff dimension less than $n-2$. From Sard's theorem it follows that the one dimensional Lebesgue measure of the critical values of u is zero. Consequently, $\partial\{u>\varepsilon\}$ is a regular surface for a.e. $\varepsilon>0$. In particular, one can choose $\varepsilon>0$ small enough to ensure that $B_{\kappa r}\cap\{u>\varepsilon\}\neq\emptyset$.

Now take any such small $\varepsilon > 0$, and consider the problem

$$\begin{cases} v_\varepsilon = u & \text{on } \partial B_r, \\ v_\varepsilon = u & \text{in } B_r \cap \Big(\{u<\varepsilon\} \cup \big(\{u>\varepsilon\} \setminus \Omega_0^+\big)\Big), \\ v_\varepsilon = \varepsilon & \text{in } B_{\kappa r} \cap \{u>\varepsilon\} \cap \Omega_0^+, \\ \Delta v_\varepsilon = 0 & \text{in } D_\varepsilon^+, \end{cases} \tag{7.1}$$

where

$$D_\varepsilon^+ = (B_r \setminus B_{\kappa r}) \cap \{u>\varepsilon\} \cap \Omega_0^+. \tag{7.2}$$

Observe that v_ε can be obtained by minimizing the Dirichlet integral over B_r subject to the constraints in (7.1). In fact, the function

$$w_\varepsilon = \begin{cases} u & \text{in } \{u<\varepsilon\} \\ \varepsilon & \text{in } \{u>\varepsilon\} \cap \Omega_0^+ \cap B_{\kappa r} \\ u & \text{elsewhere} \end{cases}$$

satisfies the boundary constraints in (7.1), and hence

$$\int_{B_r} |\nabla v_\varepsilon|^2 \leqslant \int_{B_r} |\nabla w_\varepsilon|^2 \leqslant C, \tag{7.3}$$

for some tame constant $C>0$ independent of ε. Also, v_ε is continuous at $\{u=\varepsilon\} \cap (B_r \setminus B_{\kappa r})$. We claim that

$$v_\varepsilon \leqslant u \ \text{ in } \overline{D_\varepsilon^+}. \tag{7.4}$$

Indeed, by inspection we see that $v_\varepsilon \leqslant u$ on ∂D_ε^+. Moreover, $D_\varepsilon^+ \subseteq \{u>\varepsilon\}$ and so u is harmonic there, thanks to Corollary 1.2. Hence, we obtain (7.4) from the comparison principle.

Now, formulas (7.3) and (7.4) imply that $v_\varepsilon \to v$ weakly in $W^{1,2}(B_r)$, as $\varepsilon \to 0$, and $v \leqslant u$. Furthermore, v is continuous in B_r and solves

$$\begin{cases} v = u & \text{on } \partial B_r, \\ v = u & \text{in } B_r \cap \Big(\{u\leqslant 0\} \cup \big(\{u>0\} \setminus \Omega_0^+\big)\Big), \\ v = 0 & \text{in } B_{\kappa r} \cap \Omega_0^+, \\ \Delta v = 0 & \text{in } D^+, \end{cases}$$

where

$$D^+ := (B_r \setminus B_{\kappa r}) \cap \Omega_0^+. \tag{7.5}$$

The former follows from a customary approximation argument as on page 437 of [2], and hence omitted here.

Now we extend v to be equal to u in $\Omega \setminus B_r$ and we compare $J[u]$ with $J[v]$ in Ω. Accordingly, the minimality of u gives that

$$\int_\Omega |\nabla u|^2 + \Phi_0\left(\lambda_2 \int_\Omega Q\chi_{\{u>0\}}\right) \leqslant \int_\Omega |\nabla v|^2 + \Phi_0\left(\lambda_2 \int_\Omega Q\chi_{\{v>0\}}\right).$$

This implies that

$$\int_{B_r} |\nabla u|^2 - \int_{B_r} |\nabla v|^2 = \int_\Omega |\nabla u|^2 - \int_\Omega |\nabla v|^2 \leqslant \Phi_0\left(\lambda_2 \int_\Omega Q\chi_{\{v>0\}}\right) - \Phi_0\left(\lambda_2 \int_\Omega Q\chi_{\{u>0\}}\right). \tag{7.6}$$

On the other hand, recalling that $v=u$ in $B_r \setminus \Omega_0^+$ and $v=0$ in $B_{\kappa r} \cap \Omega_0^+$, we see that

$$\begin{aligned} \int_{B_r} |\nabla u|^2 - |\nabla v|^2 &= \int_{B_r \cap \Omega_0^+} |\nabla u|^2 - |\nabla v|^2 \\ &= \int_{D^+} |\nabla u|^2 - |\nabla v|^2 + \int_{B_{\kappa r} \cap \Omega_0^+} |\nabla u|^2, \end{aligned} \tag{7.7}$$

where (7.5) was also used. This and (7.6) give that

$$\int_{D^+} |\nabla u|^2 - |\nabla v|^2 + \int_{B_{\kappa r} \cap \Omega_0^+} |\nabla u|^2 \leqslant \Phi_0\left(\lambda_2 \int_\Omega Q\chi_{\{v>0\}}\right) - \Phi_0\left(\lambda_2 \int_\Omega Q\chi_{\{u>0\}}\right). \tag{7.8}$$

Now we write

$$\begin{aligned}&\Phi_0\left(\lambda_2\int_\Omega Q\chi_{\{u>0\}}\right)-\Phi_0\left(\lambda_2\int_\Omega Q\chi_{\{v>0\}}\right)\\&=\Phi_0\left(\lambda_2\int_{\Omega\setminus B_{\kappa r}}Q\chi_{\{u>0\}}+\lambda_2\int_{B_{\kappa r}}Q\chi_{\{u>0\}}\right)-\Phi_0\left(\lambda_2\int_{\Omega\setminus B_{\kappa r}}Q\chi_{\{v>0\}}+\lambda_2\int_{B_{\kappa r}}Q\chi_{\{v>0\}}\right).\end{aligned}\tag{7.9}$$

Notice that $v=0$ in $B_{\kappa r}\cap\Omega_0^+$ and $v=u$ in $B_{\kappa r}\setminus\Omega_0^+$, hence

$$\int_{B_{\kappa r}}Q\chi_{\{v>0\}}=\int_{B_{\kappa r}\setminus\Omega_0^+}Q\chi_{\{u>0\}}.\tag{7.10}$$

Moreover, $v=u$ in $\Omega\setminus B_r$ and in $(B_r\setminus B_{\kappa r})\setminus\Omega_0^+$. Also, in $(B_r\setminus B_{\kappa r})\cap\Omega_0^+=D^+$, we have that $u>0$, by definition of Ω_0^+, and therefore

$$\chi_{\{u>0\}}\geqslant\chi_{\{v>0\}}\quad\text{in } D^+.$$

This implies that

$$\int_{\Omega\setminus B_{\kappa r}}Q\chi_{\{v>0\}}\leqslant\int_{\Omega\setminus B_{\kappa r}}Q\chi_{\{u>0\}}.\tag{7.11}$$

Plugging (7.10) and (7.11) into (7.9), and using (1.7), we obtain that

$$\begin{aligned}&\Phi_0\left(\lambda_2\int_\Omega Q\chi_{\{u>0\}}\right)-\Phi_0\left(\lambda_2\int_\Omega Q\chi_{\{v>0\}}\right)\\&\geqslant\Phi_0\left(\lambda_2\int_{\Omega\setminus B_{\kappa r}}Q\chi_{\{u>0\}}+\lambda_2\int_{B_{\kappa r}}Q\chi_{\{u>0\}}\right)-\Phi_0\left(\lambda_2\int_{\Omega\setminus B_{\kappa r}}Q\chi_{\{u>0\}}+\lambda_2\int_{B_{\kappa r}\setminus\Omega_0^+}Q\chi_{\{u>0\}}\right).\end{aligned}$$

Therefore, from the Mean Value Theorem we get

$$\begin{aligned}&\Phi_0\left(\lambda_2\int_\Omega Q\chi_{\{u>0\}}\right)-\Phi_0\left(\lambda_2\int_\Omega Q\chi_{\{v>0\}}\right)\\&\geqslant\Theta\left(\lambda_2\int_{B_{\kappa r}}Q\chi_{\{u>0\}}-\lambda_2\int_{B_{\kappa r}\setminus\Omega_0^+}Q\chi_{\{u>0\}}\right)=\Theta\lambda_2\int_{B_{\kappa r}\cap\Omega_0^+}Q\chi_{\{u>0\}},\end{aligned}\tag{7.12}$$

where

$$\Theta:=\inf_{[\lambda_2 Q_1\varpi/2,\ \lambda_2\lambda_\Omega]}\Phi_0'$$

and (1.18) has been used to estimate the interval in the definition of Θ.

So, from (7.8) and (7.12) we deduce that

$$\int_{D^+}|\nabla u|^2-|\nabla v|^2+\int_{B_{\kappa r}\cap\Omega_0^+}|\nabla u|^2\leqslant-\Theta\lambda_2\int_{B_{\kappa r}\cap\Omega_0^+}Q\chi_{\{u>0\}}.$$

Using this, we obtain that

$$\min\{1,\,\Theta\}\int_{B_{\kappa r}\cap\Omega_0^+}\left(|\nabla u|^2+\lambda_2 Q\chi_{\{u>0\}}\right)\leqslant\int_{D^+}|\nabla v|^2-|\nabla u|^2.\tag{7.13}$$

Now we observe that

$$\begin{aligned}\int_{D^+}|\nabla v|^2-|\nabla u|^2&=\int_{D^+}(\nabla v-\nabla u)\cdot(\nabla u-\nabla v+2\nabla v)\\&=-\int_{D^+}|\nabla u-\nabla v|^2+2\int_{D^+}\nabla v\cdot(\nabla v-\nabla u)\leqslant2\int_{D^+}\nabla v\cdot(\nabla v-\nabla u).\end{aligned}$$

Plugging this into (7.13), recalling that v_ε is a solution of (7.1) and using the Divergence Theorem, we obtain

$$
\begin{aligned}
&\min\{1,\,\Theta\}\int_{B_{\kappa r}\cap\Omega_0^+}\left(|\nabla u|^2+\lambda_2 Q\chi_{\{u>0\}}\right)\\
&\quad\leqslant 2\int_{D^+}\nabla v\cdot(\nabla v-\nabla u)\\
&\quad\leqslant 2\liminf_{\varepsilon\to0}\int_{D_\varepsilon^+}\nabla v_\varepsilon\cdot(\nabla v_\varepsilon-\nabla u)\\
&\quad= 2\liminf_{\varepsilon\to0}\left[\int_{D_\varepsilon^+}\operatorname{div}\left(\nabla v_\varepsilon(v_\varepsilon-u)\right)-\int_{D_\varepsilon^+}\Delta v_\varepsilon(v_\varepsilon-u)\right]\\
&\quad= 2\liminf_{\varepsilon\to0}\int_{\partial D_\varepsilon^+}\frac{\partial v_\varepsilon}{\partial\nu}\,(v_\varepsilon-u)\\
&\quad\leqslant 2\liminf_{\varepsilon\to0}\int_{\partial B_{\kappa r}\cap\{u>\varepsilon\}\cap\Omega_0^+}\left|\frac{\partial v_\varepsilon}{\partial\nu}\right|\,(u-\varepsilon).
\end{aligned}
\tag{7.14}
$$

Notice that we have performed an integration by parts, which actually needs a justification, since D_ε^+ has not smooth boundary. This can be done using an approximation of D_ε^+ by domains whose boundaries are C^∞ curves, as in [2] (see in particular page 437 there).

Our next goal is to estimate the quantity

$$2\liminf_{\varepsilon\to0}\int_{\partial B_{\kappa r}\cap\{u>\varepsilon\}\cap\Omega_0^+}\left|\frac{\partial v_\varepsilon}{\partial\nu}\right|\,(u-\varepsilon).$$

For this, we set $\kappa':=(\kappa+1)/2$ and we introduce the barrier b as follows:

$$
\begin{cases}
b=\varepsilon+\sup\limits_{\partial B_{\kappa' r}\cap\Omega_0^+} v_\varepsilon & \text{on } \partial B_{\kappa' r},\\
b=\varepsilon & \text{on } \partial B_{\kappa r},\\
\Delta b=0 & \text{in } B_{\kappa' r}\setminus B_{\kappa r}.
\end{cases}
\tag{7.15}
$$

Notice that $b\geqslant\varepsilon$ on $\partial\left(B_{\kappa' r}\setminus B_{\kappa r}\right)$ and so, by comparison principle, we have that

$$b\geqslant\varepsilon \text{ in } \overline{B_{\kappa' r}}\setminus B_{\kappa r}. \tag{7.16}$$

Recalling (7.2), we set

$$D_{\varepsilon,*}^+:=D_\varepsilon^+\cap B_{\kappa' r}=(B_{\kappa' r}\setminus B_{\kappa r})\cap\{u>\varepsilon\}\cap\Omega_0^+$$

and we claim that

$$v_\varepsilon\leqslant b \text{ on } \partial D_{\varepsilon,*}^+. \tag{7.17}$$

For this, we use the elementary formula, given sets A and B,

$$\partial(A\cap B)\subseteq\Big(\partial A\cap\overline{B}\Big)\cup\Big(\partial B\cap\overline{A}\Big). \tag{7.18}$$

This gives that

$$\partial D_{\varepsilon,*}^+\subseteq D_1\cup D_2,$$

with

$$
\begin{aligned}
&D_1:=\left(\partial B_{\kappa r}\cup\partial B_{\kappa' r}\right)\cap\{x\in\overline{\Omega_0^+}\text{ s.t. } u(x)\geqslant\varepsilon\}\cap\overline{D_\varepsilon^+}\\
\text{and}\qquad &D_2:=\left(\overline{B_{\kappa' r}}\setminus B_{\kappa r}\right)\cap\left(\partial\{x\in\Omega_0^+\text{ s.t. } u(x)>\varepsilon\}\right)\cap\overline{D_\varepsilon^+}.
\end{aligned}
$$

Now, in light of (7.15) and (7.1), if $x\in D_1$, we have that either $x\in\partial B_{\kappa' r}\cap\{u\geqslant\varepsilon\}\cap\overline{\Omega_0^+}$, and then $b(x)\geqslant\sup_{\partial B_{\kappa' r}\cap\Omega_0^+}v_\varepsilon\geqslant v_\varepsilon(x)$, or $x\in\partial B_{\kappa r}\cap\{u\geqslant\varepsilon\}\cap\overline{\Omega_0^+}$ and $b(x)=\varepsilon=v_\varepsilon(x)$. Accordingly,

$$b\geqslant v_\varepsilon \text{ in } D_1. \tag{7.19}$$

On the other hand, using again (7.18), one sees that

$$\partial\{x\in\Omega_0^+\text{ s.t. } u(x)>\varepsilon\}\ \subseteq\ (\partial\Omega_0^+)\cup(\partial\{u>\varepsilon\})\ \subseteq\ (\partial\Omega)\cup(\partial\{u>0\})\cup(\partial\{u>\varepsilon\})$$

and so

$$D_2\subseteq\left(\overline{B_{\kappa' r}}\setminus B_{\kappa r}\right)\cap\Big(\{u=0\}\cup\{u=\varepsilon\}\Big)\cap\overline{D_\varepsilon^+}.$$

As a consequence, if $x \in D_2$, then $x \in \overline{B_{\kappa' r}} \setminus B_{\kappa r}$ and $u(x) \leqslant \varepsilon$. This and (7.16) give that $u \leqslant b$ in D_2. Hence, in view of (7.4), we find that $v_\varepsilon \leqslant b$ in D_2. This, together with (7.19), proves (7.17).

Now, by (7.1), (7.15), (7.17) and the comparison principle, we conclude that $v_\varepsilon \leqslant b$ in $D^+_{\varepsilon,*}$. Therefore, since $v_\varepsilon = \varepsilon = b$ on $\partial B_{\kappa r} \cap \{u > \varepsilon\} \cap \Omega_0^+$, we find that

$$|\nabla v_\varepsilon| \leqslant |\nabla b| \text{ on } \partial B_{\kappa r} \cap \{u > \varepsilon\} \cap \Omega_0^+. \tag{7.20}$$

As a matter of fact, we can explicitly solve b in (7.15), and we have that

$$b(x) = \frac{\sup_{\partial B_{\kappa' r}\cap\Omega_0^+} v_\varepsilon}{\Psi(\kappa r) - \Psi(\kappa' r)} \big(\Psi(\kappa' r) - \Psi(|x|)\big) + \varepsilon + \sup_{\partial B_{\kappa' r}\cap\Omega_0^+} v_\varepsilon,$$

where Ψ is the radially decreasing fundamental solution of the Laplace operator in $\mathbb{R}^n$ (up to normalizing constants, $\Psi(\rho) = \rho^{2-n}$ if $n \geqslant 3$ and $\Psi(\rho) = -\log\rho$ if $n = 2$).

Consequently, recalling also (7.4),

$$\sup_{\partial B_{\kappa r}} |\nabla b| = C \frac{\sup_{\partial B_{\kappa' r}\cap\Omega_0^+} v_\varepsilon}{\Psi(\kappa r) - \Psi(\kappa' r)} |\nabla \Psi(\kappa r)| \leqslant C \frac{\sup_{\partial B_{\kappa' r}\cap\Omega_0^+} u}{r}, \tag{7.21}$$

for some $C > 0$ possibly depending on κ and different from step to step.

Now, we observe that, extending u by zero outside Ω_0^+, we obtain a nonnegative subharmonic function. More precisely, if we set $\tilde u := u\chi_{\Omega_0^+}$, given any nonnegative $\phi \in C_0^\infty(\Omega)$, we have that

$$\begin{aligned}\int_\Omega \tilde u\,\Delta\phi &= \lim_{\varepsilon\to0} \int_{\Omega_0^+\cap\{u>\varepsilon\}} u\,\Delta\phi = \lim_{\varepsilon\to0}\int_{\Omega_0^+\cap\{u>\varepsilon\}} \Big(\phi\,\Delta u + \operatorname{div}(u\nabla\phi - \phi\nabla u)\Big)\\ &= 0 + \lim_{\varepsilon\to0}\int_{\overline{\Omega_0^+}\cap\partial\{u>\varepsilon\}} u\frac{\partial\phi}{\partial\nu} - \phi\frac{\partial u}{\partial\nu} = -\lim_{\varepsilon\to0}\int_{\overline{\Omega_0^+}\cap\partial\{u>\varepsilon\}} \phi\frac{\partial u}{\partial\nu} \geqslant 0,\end{aligned}$$

and so $\tilde u$ is subharmonic. Hence the weak maximum principle can be applied to the function $\tilde u$, and so we conclude that, for any $\sigma \in (0,1)$ and any $x \in B_{\sigma r}$,

$$\begin{aligned} u(x)\chi_{\Omega_0^+}(x) = \tilde u(x) &\leqslant \fint_{B_{(1-\sigma)r}(x)} \tilde u \leqslant \left(\fint_{B_{(1-\sigma)r}(x)} \tilde u^2\right)^{\frac12} \leqslant \left(\frac{1}{\mathcal{L}^n(B_{(1-\sigma)r})}\int_{B_r}\tilde u^2\right)^{\frac12}\\ &\leqslant \left(\frac{\mathcal{L}^n(B_r\cap\Omega_0^+)}{\mathcal{L}^n(B_{(1-\sigma)r})}\fint_{B_r\cap\Omega_0^+} u^2\right)^{\frac12} \leqslant \left(\frac{r^n}{(1-\sigma)^n r^n}\fint_{B_r\cap\Omega_0^+} u^2\right)^{\frac12}.\end{aligned}$$

Thus,

$$\sup_{B_{\sigma r}\cap\Omega_0^+} u \leqslant \frac{1}{(1-\sigma)^{\frac n2}}\left(\fint_{B_r\cap\Omega_0^+} u^2\right)^{\frac12} = \frac{r\gamma}{(1-\sigma)^{\frac n2}}, \tag{7.22}$$

where we have set

$$\gamma := \left(\frac{1}{r^2}\fint_{B_r\cap\Omega_0^+} u^2\right)^{\frac12}.$$

From this and (7.21) we conclude that

$$\sup_{\partial B_{\kappa r}} |\nabla b| \leqslant C\gamma. \tag{7.23}$$

From (7.20) and (7.23), it follows that

$$\begin{aligned}\int_{\partial B_{\kappa r}\cap\{u>\varepsilon\}\cap\Omega_0^+} \left|\frac{\partial v_\varepsilon}{\partial\nu}\right|(u-\varepsilon) &\leqslant \int_{\partial B_{\kappa r}\cap\{u>\varepsilon\}\cap\Omega_0^+} |\nabla v_\varepsilon|\,(u-\varepsilon)\\ &\leqslant \int_{\partial B_{\kappa r}\cap\{u>\varepsilon\}\cap\Omega_0^+} |\nabla b|\,(u-\varepsilon) \leqslant C\gamma\int_{\partial B_{\kappa r}\cap\{u>\varepsilon\}\cap\Omega_0^+}(u-\varepsilon)\\ &\leqslant C\gamma\int_{\partial B_{\kappa r}\cap\Omega_0^+} u.\end{aligned}$$

So, making use of (7.14), we obtain

$$\int_{B_{\kappa r}\cap\Omega_0^+}\big(|\nabla u|^2+\lambda_2 Q\chi_{\{u>0\}}\big)\leqslant C\gamma\int_{\partial B_{\kappa r}\cap\Omega_0^+}u. \tag{7.24}$$

Now we recall the elementary trace inequality for nonnegative functions f, see e.g. Theorem 1(ii) on page 258 of [11], namely

$$\int_{\partial B_\rho} f\leqslant C\left[\int_{B_\rho}|\nabla f|+\frac{1}{\rho}\int_{B_\rho} f\right]. \tag{7.25}$$

We apply (7.25) to $f:=u\chi_{\Omega_0^+}$ and $\rho:=\kappa r$. Then, since u vanishes along $\partial\Omega_0^+$,

$$\int_{\partial B_{\kappa r}\cap\Omega_0^+}u\leqslant C\left[\int_{B_{\kappa r}\cap\Omega_0^+}|\nabla u|+\frac{1}{r}\int_{B_{\kappa r}\cap\Omega_0^+}u\right],$$

where $C>0$ now may also depend on κ.

Hence, in light of (7.24), we find that

$$\int_{B_{\kappa r}\cap\Omega_0^+}\big(|\nabla u|^2+\lambda_2 Q\chi_{\{u>0\}}\big)\leqslant C\gamma\left[\int_{B_{\kappa r}\cap\Omega_0^+}|\nabla u|+\frac{1}{r}\int_{B_{\kappa r}\cap\Omega_0^+}u\right]. \tag{7.26}$$

Now we point out that, by the Cauchy-Schwarz inequality,

$$2\int_{B_{\kappa r}\cap\Omega_0^+}|\nabla u|\leqslant\int_{B_{\kappa r}\cap\Omega_0^+}\big(|\nabla u|^2+1\big).$$

This and (7.26) give that

$$\begin{aligned}
&\int_{B_{\kappa r}\cap\Omega_0^+}\big(|\nabla u|^2+\lambda_2 Q\chi_{\{u>0\}}\big)\\
\leqslant\;& C\gamma\int_{B_{\kappa r}\cap\Omega_0^+}\big(|\nabla u|^2+\chi_{\{u>0\}}\big)+\frac{C\gamma}{r}\sup_{B_{\kappa r}\cap\Omega_0^+}u\int_{B_{\kappa r}\cap\Omega_0^+}\chi_{\{u>0\}}\\
\leqslant\;& C\gamma\max\left\{1,\,\frac{1}{Q_1\lambda_2}\right\}\left[\int_{B_{\kappa r}\cap\Omega_0^+}\big(|\nabla u|^2+\lambda_2 Q\chi_{\{u>0\}}\big)+\frac{1}{r}\sup_{B_{\kappa r}\cap\Omega_0^+}u\int_{B_{\kappa r}\cap\Omega_0^+}Q\chi_{\{u>0\}}\right].
\end{aligned}$$

In consequence of this and (7.22), we obtain that

$$\int_{B_{\kappa r}\cap\Omega_0^+}\big(|\nabla u|^2+\lambda_2 Q\chi_{\{u>0\}}\big)\leqslant C\gamma(1+\gamma)\int_{B_{\kappa r}\cap\Omega_0^+}\big(|\nabla u|^2+\lambda_2 Q\chi_{\{u>0\}}\big).$$

If γ is sufficiently small we conclude that u vanishes identically in $B_{\kappa r}\cap\Omega_0^+$, as desired. ☐

8. Density theorems and clean ball conditions

In this section we prove that the positive phase $\{u>0\}$ occupies a positive density near the free boundary points.

Theorem 8.1. *Assume that u is a minimizer of J as in (1.8). Let $\varpi>0$ and assume that $\mathcal{L}^n\big(\Omega^+(u)\big)\geqslant\varpi$. Let $D\Subset\Omega$. Assume that $x_0\in\Gamma=\partial\Omega^+(u)$ and let $r>0$ be such that $B_r(x_0)\subseteq D$.*

Then, there exist $c_1\in(0,1)$, possibly depending on ϖ, Q, Ω and D, and $y_0\in B_r(x_0)$ such that

$$B_{c_1 r}(y_0)\subseteq B_r(x_0)\cap\Omega^+(u). \tag{8.1}$$

Moreover, there exists $c_2>0$, possibly depending on ϖ, Q, Ω and D, such that

$$\mathcal{L}^n\big(B_r(x_0)\cap\Omega^+(u)\big)\geqslant c_2 r^n. \tag{8.2}$$

Proof. Obviously, we have that (8.2) is a direct consequence of (8.1), so we focus on the proof of (8.1). To this aim, we recall that u is continuous, thanks to Corollary 1.2, hence we can take $y_0\in\overline{B_{r/2}(x_0)}$ such that

$$u(y_0)=\max_{\overline{B_{r/2}(x_0)}}u. \tag{8.3}$$

We take $d:=\operatorname{dist}(y_0,\Gamma)$ and $z_0\in\Gamma\cap\partial B_d(y_0)$. Notice that, since $x_0\in\Gamma$, we have that

$$d\leqslant|x_0-y_0|\leqslant\frac{r}{2}. \tag{8.4}$$

Hence, we are in the position of applying Corollary 5.2, and we obtain that

$$u(y_0) \leqslant Cd, \tag{8.5}$$

for some $C > 0$. On the other hand, from Theorem 1.4 and (8.3),

$$cr^2 \leqslant \fint_{B_{r/2}(x_0)} u^2 \leqslant u^2(y_0),$$

for some $c > 0$. Comparing this with (8.5), we conclude that

$$d \geqslant \frac{u(y_0)}{C} \geqslant \frac{\sqrt{c}r}{C} = c_1\, r,$$

for some $c_1 > 0$. As a matter of fact, by (8.4), we have that $c_1 \in (0,\, 1/2)$. This construction establishes (8.1). $\square$

9. Density theorems from above

The goal of this section is to establish a counterpart of the density estimate in (8.2), by proving that the phase $\{u \leqslant 0\}$ also occupies a positive density near the free boundary points. This will be accomplished in Theorem 9.2. To this aim, we will also rely on a concavity assumption on the nonlinearity (recall (1.15)).

We need first the following auxiliary result:

Lemma 9.1. *Assume that u is a minimizer in Ω of J as in* (1.8)*. Let v_0 be such that $v_0 \geqslant u$ on $\partial\Omega$.*

Then, there exists a minimizer v of the functional J with $v = v_0$ on $\partial\Omega$ and such that $v \geqslant u$ in Ω.

Proof. We take w to be a minimizer in Ω of the functional J with $w = v_0$ on $\partial\Omega$, whose existence is guaranteed by Lemma 2.1. We set $m(x) := \min\{u(x), w(x)\}$ and $M(x) := \max\{u(x), w(x)\}$. We want to show that also M is a minimizer of J in Ω, with respect to its own boundary data along $\partial\Omega$. Once we check this, we can take $v := M$ and the desired result is established.

For this, we observe that, for any a, b, α, $\beta \geqslant 0$, we have that

$$\Phi_0(a+\alpha) + \Phi_0(a+\beta) \geqslant \Phi_0(a+\alpha+\beta) + \Phi_0(a). \tag{9.1}$$

Indeed, if we set

$$\Psi(\beta) := \Phi_0(a+\alpha) + \Phi_0(a+\beta) - \Phi_0(a+\alpha+\beta) - \Phi_0(a),$$

we have that $\Psi(0) = 0$ and

$$\Psi'(\beta) = \Phi_0'(a+\beta) - \Phi_0'(a+\alpha+\beta) \geqslant 0,$$

because Φ_0' is decreasing (due to the concavity assumption on Φ_0). Consequently, for any $\beta \geqslant 0$, we have that $\Psi(\beta) \geqslant \Psi(0) = 0$, and this proves (9.1).

Now, using the minimality of u and w, we have that $J[u] \leqslant J[m]$ and $J[w] \leqslant J[M]$. Since we want to prove the minimality of M, our goal is to check that $J[w] = J[M]$. So we argue by contradiction and we suppose that $J[w] + \delta \leqslant J[M]$, for some $\delta > 0$. Then, we have

$$\begin{aligned}
& \int_\Omega |\nabla u|^2 + |\nabla w|^2 + \Phi_0\big(\mathcal{M}_2(u)\big) + \Phi_0\big(\mathcal{M}_2(w)\big) \\
= \;& J[u] + J[w] \\
\leqslant \;& J[m] + J[M] - \delta \\
= \;& \int_\Omega |\nabla m|^2 + |\nabla M|^2 + \Phi_0\big(\mathcal{M}_2(m)\big) + \Phi_0\big(\mathcal{M}_2(M)\big) - \delta \\
= \;& \int_\Omega |\nabla u|^2 + |\nabla w|^2 + \Phi_0\big(\mathcal{M}_2(m)\big) + \Phi_0\big(\mathcal{M}_2(M)\big) - \delta,
\end{aligned}$$

that is

$$\delta \leqslant \Phi_0\big(\mathcal{M}_2(m)\big) + \Phi_0\big(\mathcal{M}_2(M)\big) - \Phi_0\big(\mathcal{M}_2(u)\big) - \Phi_0\big(\mathcal{M}_2(w)\big). \tag{9.2}$$

Now we set

$$\begin{aligned} & a := \mathcal{M}_2(m), \\ & \alpha := \mathcal{M}_2(u) - a = \mathcal{M}_2(u) - \mathcal{M}_2(m) = \lambda_2 \int_\Omega Q\, \chi_{\{u>0\geqslant w\}} \\ \text{and} \qquad & \beta := \mathcal{M}_2(w) - a = \mathcal{M}_2(w) - \mathcal{M}_2(m) = \lambda_2 \int_\Omega Q\, \chi_{\{w>0\geqslant u\}}. \end{aligned}$$

Notice that

$$a + \alpha + \beta = \lambda_2 \int_\Omega Q \left(\chi_{\{m>0\}} + \chi_{\{u>0\geqslant w\}} + \chi_{\{w>0\geqslant u\}} \right) = \lambda_2 \int_\Omega Q\, \chi_{\{M>0\}} = \mathcal{M}_2(M).$$

Consequently, (9.1) gives that

$$\Phi_0(\mathcal{M}_2(u)) + \Phi_0(\mathcal{M}_2(w)) \geqslant \Phi_0(\mathcal{M}_2(M)) + \Phi_0(\mathcal{M}_2(m)).$$

By inserting this into (9.2), we obtain that $\delta \leqslant 0$, which is a contradiction. $\square$

Theorem 9.2. *Assume that u is a minimizer in Ω for the functional J in* (1.8). *Let*

$$S := \sup_{r \in [0, \lambda_2 Q_2 \mathcal{L}^n(\Omega)]} \Phi_0'(r) < +\infty. \tag{9.3}$$

Let $D \Subset \Omega$. Assume that $x_0 \in \Gamma = \partial\Omega^+(u)$ and let $r > 0$ be such that $B_r(x_0) \subseteq D$. Then, there exists $c > 0$, possibly depending on S, Q, Ω and D, such that

$$\mathcal{L}^n\big(B_r(x_0) \cap \{u \leqslant 0\}\big) \geqslant c\, r^n. \tag{9.4}$$

Proof. First of all, we prove the desired result under the additional assumption that

$$u \geqslant 0 \text{ in } \Omega. \tag{9.5}$$

In this case, we take w to be the harmonic function in $B_r(x_0)$ such that $w = u$ in $\Omega \setminus B_r(x_0)$. Then, we set

$$\begin{aligned} & a := \lambda_2 \int_{\Omega \setminus B_r(x_0)} Q\chi_{\{u>0\}} = \lambda_2 \int_{\Omega \setminus B_r(x_0)} Q\chi_{\{w>0\}} \\ \text{and} \qquad & b := \lambda_2 \int_{B_r(x_0)} Q, \end{aligned}$$

and we have

$$\begin{aligned} 0 &\leqslant J[w] - J[u] \\ &= \int_\Omega (|\nabla w|^2 - |\nabla u|^2) + \Phi_0\left(\lambda_2 \int_\Omega Q\chi_{\{w>0\}}\right) - \Phi_0\left(\lambda_2 \int_\Omega Q\chi_{\{u>0\}}\right) \\ &= \int_{B_r(x_0)} \Big(-|\nabla(w-u)|^2 - 2\nabla w \cdot \nabla(u-w) \Big) \\ &\qquad + \Phi_0\left(a + \lambda_2 \int_{B_r(x_0)} Q\chi_{\{w>0\}}\right) - \Phi_0\left(a + \lambda_2 \int_{B_r(x_0)} Q\chi_{\{u>0\}}\right) \\ &= -\int_{B_r(x_0)} |\nabla(w-u)|^2 + \Phi_0\left(a + b - \lambda_2 \int_{B_r(x_0)} Q\chi_{\{w\leqslant 0\}}\right) - \Phi_0\left(a + b - \lambda_2 \int_{B_r(x_0)} Q\chi_{\{u\leqslant 0\}}\right) \end{aligned}$$

and so, by Poincaré inequality,

$$\begin{aligned} & \Phi_0(a+b) - \Phi_0\left(a + b - \lambda_2 \int_{B_r(x_0)} Q\chi_{\{u\leqslant 0\}}\right) \\ & \geqslant \Phi_0\left(a + b - \lambda_2 \int_{B_r(x_0)} Q\chi_{\{w\leqslant 0\}}\right) - \Phi_0\left(a + b - \lambda_2 \int_{B_r(x_0)} Q\chi_{\{u\leqslant 0\}}\right) \\ & \geqslant \int_{B_r(x_0)} |\nabla(w-u)|^2 \\ & \geqslant \frac{c}{r^2} \int_{B_r(x_0)} |w-u|^2, \end{aligned} \tag{9.6}$$

for some $c > 0$.

On the other hand,

$$\Phi_0\,(a+b)-\Phi_0\left(a+b-\lambda_2\int_{B_r(x_0)}Q\chi_{\{u\leqslant 0\}}\right)\leqslant S\lambda_2\int_{B_r(x_0)}Q\chi_{\{u\leqslant 0\}}\leqslant S\lambda_2 Q_2\,\mathcal{L}^n\big(B_r(x_0)\cap\{u\leqslant 0\}\big). \tag{9.7}$$

Also, since u is subharmonic (by Lemma 2.2), we have that $w\geqslant u$ in $B_r(x_0)$. We now fix $\kappa>0$, to be taken suitably small in the sequel. By Corollary 5.2, we have that $\sup_{B_{\kappa r}(x_0)}u\leqslant C\kappa r$, for some $C>0$.

In addition, by (8.1), we have that there exists $y_0\in B_{r/2}(x_0)$ such that

$$B_{c_1r}(y_0)\subseteq\{u>0\},$$

for some $c_1>0$, hence the distance of y_0 from the free boundary is at least c_1r. This and Lemma 5.4 give that $u(y_0)\geqslant c_2r$, for some $c_2>0$.

By Corollary 1.2, u is harmonic in $B_{c_1r}(y_0)$, thus the Harnack inequality implies that $u\geqslant c_3r$ in $B_{c_1r/2}(y_0)$, for some $c_3>0$. Hence, by the harmonicity of w,

$$w(x_0)=\fint_{B_r(x_0)}w\geqslant\fint_{B_r(x_0)}u\geqslant\frac{1}{\mathcal{L}^n(B_r)}\int_{B_{c_1r/2}(y_0)}u\geqslant c_4r,$$

for some $c_4>0$. Consequently, by Harnack inequality, we obtain that

$$\inf_{B_{\kappa r}(x_0)}w\geqslant\bar c\,r,$$

for some $\bar c>0$.

From these considerations, we obtain that, in $B_{\kappa r}(x_0)$,

$$|w-u|=w-u\geqslant(\bar c-C\kappa)r\geqslant\frac{\bar c r}{2},$$

as long as κ is small enough. Accordingly,

$$\int_{B_r(x_0)}|w-u|^2\geqslant\int_{B_{\kappa r}(x_0)}|w-u|^2\geqslant\tilde c r^{n+2},$$

for some $\tilde c>0$.

We insert this and (9.7) into (9.6) and we conclude that

$$S\lambda_2Q_2\,\mathcal{L}^n\big(B_r(x_0)\cap\{u\leqslant 0\}\big)\geqslant\hat c r^n,$$

for some $\hat c>0$. This proves the desired result under the additional assumption in (9.5).

To deal with the general case, we use Lemma 9.1: namely, we take v which is a minimizer in Ω for the functional J in (1.8), with $v=u^+$ on $\partial\Omega$ and such that $v\geqslant u$ in Ω. From the fact that $v\geqslant 0$ on $\partial\Omega$, we deduce that $v\geqslant 0$ in Ω (see e.g. Lemma 2.3 in [1]). Thus, since (9.4) has been established for nonnegative minimizers, we know that

$$\mathcal{L}^n\big(B_r(x_0)\cap\{v\leqslant 0\}\big)\geqslant c\,r^n.$$

Moreover, since $v\geqslant u$, we have that $\{u\leqslant 0\}\supseteq\{v\leqslant 0\}$, and so (9.4) follows for u. □

10. Blow-up limits

In this section, we consider the blow-up of a minimizer at a free boundary point. We will show that, in the limit, we obtain a minimizer for the Alt-Caffarelli problem in (1.11). This phenomenon plays an important role in our analysis, since it transforms the original nonlinear free boundary problem into a linear one, in the blow-up limit: that is, in our framework, the blow-up possesses an additional linearization feature.

To this extent, for any $x_0\in\Gamma$ we consider the blow-up sequence of u at x_0, that is

$$u_k(x):=\frac{u(x_0+\rho_kx)}{\rho_k}, \tag{10.1}$$

where $\rho_k\to 0$ as $k\to+\infty$.

We have the following convergence result (see e.g. Proposition 8.1 in [9] for the proof):

Proposition 10.1. *Let $x_0 \in \Gamma$ and u_k a the blow-up sequence, as introduced in* (10.1)*.*

Then there exists a blow-up limit $u_0 : \mathbb{R}^n \to \mathbb{R}$, which is continuous and with linear growth, such that, up to a subsequence, as $k \to +\infty$,

- $u_k \to u_0$ *in $C^{\alpha}_{loc}(\mathbb{R}^n)$ for any $\alpha \in (0,1)$,*
- $\nabla u_k \to \nabla u_0$ *weakly in $L^q_{loc}(\mathbb{R}^n)$ for any $q > 1$,*
- $\partial\{u_k > 0\} \to \partial\{u_0 > 0\}$ *locally in Hausdorff distance,*
- $\chi_{\{u_k>0\}} \to \chi_{\{u_0>0\}}$ *in $L^1_{loc}(\mathbb{R}^n)$.*

The statement above can be also enhanced, giving the pointwise convergence of the gradients, as given by the next result:

Lemma 10.2. *Let $x_0 \in \Gamma$. Let u_k be the blow-up sequence, as introduced in* (10.1)*, and u_0 the blow-up limit given by Proposition* 10.1*. Then $\nabla u_k \to \nabla u_0$ a.e. in $\mathbb{R}^n$, as $k \to +\infty$.*

In addition, if $p \in \{u_0 \neq 0\}$, we have that $\nabla u_k \to \nabla u_0$ as $k \to +\infty$ uniformly in a neighborhood of p.

Proof. The proof is an appropriate modification (and actually a simplification) of some arguments also exploited in [1]. We let $\mathscr{A}$ be the set of the Lebesgue density points of $\{u_0 = 0\}$. We show that $\nabla u_k \to \nabla u_0$ in $\mathscr{A} \cup \{u_0 \neq 0\}$, as $k \to +\infty$, with locally uniform convergence in $\{u_0 \neq 0\}$ (with this, since the complement of $\mathscr{A} \cup \{u_0 \neq 0\}$ has zero Lebesgue measure, the desired result is established).

To this aim, we observe that if $p \in \{u_0 \neq 0\}$ we know from Proposition 10.1 that there exists $r_0 > 0$ such that $u_k(x) \neq 0$ for any $x \in B_{r_0}(p)$, as long as k is large enough. Then, by Corollary 1.2, we have that u_k is harmonic in $B_{r_0}(p)$ and so it has second derivatives estimates in $B_{r_0/2}(p)$. This implies that $\nabla u_k \to \nabla u_0$ uniformly in $B_{r_0/2}(p)$ and so, in particular, that $\nabla u_k(p) \to \nabla u_0(p)$, as $k \to +\infty$.

Now, let us take $q \in \mathscr{A}$. Then,

$$\lim_{r \searrow 0} \frac{\mathcal{L}^n(B_r(q) \cap \{u_0 = 0\})}{\mathcal{L}^n(B_r(q))} = 1$$

and therefore for any $\eta > 0$ there exists $\bar{r}(\eta) > 0$ such that if $r \in (0, \bar{r}(\eta)]$ then

$$\frac{\mathcal{L}^n(B_r(q) \cap \{u_0 = 0\})}{\mathcal{L}^n(B_r(q))} \geqslant 1 - \eta.$$

In particular $\mathcal{L}^n(B_r(q) \cap \{u_0 \neq 0\}) \leqslant \eta\, \mathcal{L}^n(B_r(q))$ and so, in light of the Lipschitz regularity obtained in Theorem 1.3, we have that

$$\fint_{B_r(q)} u_0^2 = \frac{1}{\mathcal{L}^n(B_r(q))} \int_{B_r(q) \cap \{u_0 \neq 0\}} u_0^2 \leqslant C r^2 \frac{\mathcal{L}^n(B_r(q) \cap \{u_0 \neq 0\})}{\mathcal{L}^n(B_r(q))} \leqslant C\eta\, r^2 < \frac{c}{2} r^2,$$

up to renaming $C > 0$ and taking η suitably small, where $c > 0$ is the one given in Theorem 1.4. Consequently

$$\fint_{B_r(q)} u_k^2 < c\, r^2$$

if k is large enough, and so, by Theorem 1.4, we have that $u_k \leqslant 0$ in $B_r(q)$, and so $u_0 \leqslant 0$ in $B_r(q)$.

We also know that u_k is subharmonic, thanks to Lemma 2.2, and thus also u_0 is subharmonic. Accordingly, for small $r > 0$,

$$0 = u_0(q) \leqslant \fint_{B_r(q)} u_0 \leqslant 0,$$

which implies that u_0 vanishes identically in $B_r(q)$. Similarly, u_k vanishes identically in $B_r(q)$. These considerations imply that $\nabla u_k(q) = 0 = \nabla u_0(q)$. ☐

Next result shows that the blow-up limit u_0 is always a minimizer of the Alt-Caffarelli functional in (1.11) (for a suitable choice of Q, which turns out to be constant). That is, the blow-up limit has the additional, and somehow unexpected advantage, to linearize the interfacial energy. The precise result, which was stated in Theorem 1.5, is proved by the following argument:

Proof of Theorem 1.5. The result in Theorem 1.5 can be seen as a nonlinear counterpart of Lemma 5.4 in [1]. Up to a translation, we take $x_0 := 0$ in (10.1). We take a competitor v_0 for u_0, i.e. we suppose that $v_0 - u_0 \in W^{1,2}_0(B_r)$. We also take $\eta \in C^\infty_0(B_r,\,[0,1])$ and we define

$$v_\rho := v_0 + (1-\eta)(u_\rho - u_0).$$

We observe that

$$v_\rho - u_\rho = (v_0 - u_0) - \eta(u_\rho - u_0) \tag{10.2}$$

and so

$$v_\rho - u_\rho = 0 \text{ outside } B_r. \tag{10.3}$$

In addition,

$$\{v_\rho > 0\} \subseteq \{v_0 > 0\} \cup \{\eta < 1\}$$

and therefore

$$\chi_{\{v_\rho>0\}} \leqslant \chi_{\{v_0>0\}} + \chi_{\{\eta<1\}}. \tag{10.4}$$

We also define $v(x) := \rho v_\rho(x/\rho)$. We remark that

$$(v-u)(x) = \rho\left(v_\rho\left(\frac{x}{\rho}\right) - u_\rho\left(\frac{x}{\rho}\right)\right) = 0 \quad \text{for any } x \in \mathbb{R}^n \setminus B_{\rho r},$$

thanks to (10.1) and (10.3). Since $B_{\rho r} \subset \Omega$ when ρ is sufficiently small (possibly in dependence of the fixed $r > 0$), we obtain that $v - u = 0$ outside Ω.

Consequently, we can use the minimality of u in Ω and find that

$$\begin{aligned} 0 \leqslant J[v] - J[u] &= \int_\Omega \left(|\nabla v|^2 - |\nabla u|^2\right) + \Phi_0\left(\lambda_2 \int_\Omega Q\chi_{\{v>0\}}\right) - \Phi_0\left(\lambda_2 \int_\Omega Q\chi_{\{u>0\}}\right) \\ &= \int_{B_{\rho r}} \left(|\nabla v|^2 - |\nabla u|^2\right) + \Phi_0\left(\lambda_2 \int_{B_{\rho r}} Q\chi_{\{v>0\}} + \Xi_\rho\right) - \Phi_0\left(\lambda_2 \int_{B_{\rho r}} Q\chi_{\{u>0\}} + \Xi_\rho\right), \end{aligned} \tag{10.5}$$

where

$$\Xi_\rho := \lambda_2 \int_{\Omega\setminus B_{\rho r}} Q\chi_{\{u>0\}}.$$

We point out that

$$\lim_{\rho\searrow 0} \Xi_\rho = \Xi_0 := \lambda_2 \int_\Omega Q\chi_{\{u>0\}}\,. \tag{10.6}$$

Now we scale the quantities in (10.5), using the substitution $y := x/\rho$. In this way, we find that

$$\int_{B_{\rho r}} |\nabla v(x)|^2\,dx = \int_{B_{\rho r}} |\nabla v_\rho(x/\rho)|^2\,dx = \rho^n \int_{B_r} |\nabla v_\rho(y)|^2\,dy$$

and

$$\int_{B_{\rho r}} Q(x)\chi_{\{v>0\}}(x)\,dx = \int_{B_{\rho r}} Q(x)\chi_{\{v_\rho>0\}}(x/\rho)\,dx = \rho^n \int_{B_r} Q(\rho y)\chi_{\{v_\rho>0\}}(y)\,dy,$$

and similar expressions hold true with u replacing v. Substituting these identities into (10.5), we conclude that

$$\begin{aligned} 0 \leqslant &\int_{B_r} \left(|\nabla v_\rho|^2 - |\nabla u_\rho|^2\right) \\ &+ \frac{1}{\rho^n}\left[\Phi_0\left(\rho^n\lambda_2 \int_{B_r} Q(\rho x)\chi_{\{v_\rho>0\}}(x)\,dx + \Xi_\rho\right) - \Phi_0\left(\rho^n\lambda_2 \int_{B_r} Q(\rho x)\chi_{\{u_\rho>0\}}(x)\,dx + \Xi_\rho\right)\right]. \end{aligned} \tag{10.7}$$

Now, from (10.2), we have that

$$v_\rho + u_\rho = v_0 - u_0 - \eta(u_\rho - u_0) + 2u_\rho = (v_0 + u_0) + (2-\eta)(u_\rho - u_0).$$

This and (10.2) give that

$$\begin{aligned}
|\nabla v_\rho|^2 - |\nabla u_\rho|^2 &= \nabla(v_\rho + u_\rho)\cdot(v_\rho - u_\rho)\\
&= \nabla\big((v_0+u_0)+(2-\eta)(u_\rho-u_0)\big)\cdot\nabla\big((v_0-u_0)-\eta(u_\rho-u_0)\big)\\
&= \nabla(v_0+u_0)\cdot\nabla(v_0-u_0) - \eta\nabla(v_0+u_0)\cdot\nabla(u_\rho-u_0) - (u_\rho-u_0)\nabla(v_0+u_0)\cdot\nabla\eta\\
&\qquad -(u_\rho-u_0)\nabla\eta\cdot\nabla(v_0-u_0) + (2-\eta)\nabla(u_\rho-u_0)\cdot\nabla(v_0-u_0)\\
&\qquad +|u_\rho-u_0|^2\,|\nabla\eta|^2 + \eta(u_\rho-u_0)\nabla\eta\cdot\nabla(u_\rho-u_0)\\
&\qquad -(2-\eta)(u_\rho-u_0)\nabla(u_\rho-u_0)\cdot\nabla\eta - (2-\eta)\eta\,|\nabla(u_\rho-u_0)|^2.
\end{aligned}$$

We remark that the latter term has a sign. So, recalling Proposition 10.1, we obtain that

$$\lim_{\rho\searrow 0}\int_{B_r}\big(|\nabla v_\rho|^2 - |\nabla u_\rho|^2\big) \leqslant \int_{B_r}\nabla(v_0+u_0)\cdot\nabla(v_0-u_0) = \int_{B_r}\big(|\nabla v_0|^2-|\nabla u_0|^2\big). \tag{10.8}$$

Now we set

$$\begin{aligned}
&\alpha_\rho := \lambda_2\int_{B_r}Q(\rho x)\chi_{\{v_0>0\}}(x)\,dx + \lambda_2\int_{B_r}Q(\rho x)\chi_{\{\eta<1\}}(x)\,dx\\
\text{and}\qquad &\beta_\rho := \lambda_2\int_{B_r}Q(\rho x)\chi_{\{u_\rho>0\}}(x)\,dx.
\end{aligned}$$

We observe that

$$\begin{aligned}
&\lim_{\rho\searrow 0}\alpha_\rho = \alpha_0 := \lambda_2\int_{B_r}Q(0)\chi_{\{v_0>0\}}(x)\,dx + \lambda_2\int_{B_r}Q(0)\chi_{\{\eta<1\}}(x)\,dx\\
\text{and}\qquad &\lim_{\rho\searrow 0}\beta_\rho = \beta_0 := \lambda_2\int_{B_r}Q(0)\chi_{\{u_0>0\}}(x)\,dx,
\end{aligned}$$

thanks to Proposition 10.1.

Then, recalling (10.4) and the monotonicity of Φ_0, and exploiting also (10.6), we have that

$$\begin{aligned}
&\lim_{\rho\searrow 0}\frac{1}{\rho^n}\left[\Phi_0\left(\rho^n\lambda_2\int_{B_r}Q(\rho x)\chi_{\{v_\rho>0\}}(x)\,dx+\Xi_\rho\right) - \Phi_0\left(\rho^n\lambda_2\int_{B_r}Q(\rho x)\chi_{\{u_\rho>0\}}(x)\,dx+\Xi_\rho\right)\right]\\
\leqslant\ &\lim_{\rho\searrow 0}\frac{1}{\rho^n}\Big[\Phi_0(\rho^n\alpha_\rho+\Xi_\rho)-\Phi_0(\rho^n\beta_\rho+\Xi_\rho)\Big]\\
=\ &\lim_{\rho\searrow 0}(\alpha_\rho-\beta_\rho)\int_0^1\Phi_0'\big(\rho^n\beta_\rho+t\rho^n(\alpha_\rho-\beta_\rho)+\Xi_\rho\big)\,dt\\
=\ &(\alpha_0-\beta_0)\,\Phi_0'(\Xi_0).
\end{aligned}$$

So, we insert this inequality and (10.8) into (10.7) and we obtain

$$\begin{aligned}
0\ &\leqslant\ \int_{B_r}\big(|\nabla v_0|^2-|\nabla u_0|^2\big) + (\alpha_0-\beta_0)\,\Phi_0'(\Xi_0)\\
&=\ \int_{B_r}\big(|\nabla v_0|^2-|\nabla u_0|^2\big) + \lambda_2 Q(0)\,\Phi_0'(\Xi_0)\int_{B_r}\chi_{\{v_0>0\}}(x)\,dx\\
&\qquad +\lambda_2 Q(0)\,\Phi_0'(\Xi_0)\int_{B_r}\chi_{\{\eta<1\}}(x)\,dx - \lambda_2 Q(0)\,\Phi_0'(\Xi_0)\int_{B_r}\chi_{\{u_0>0\}}(x)\,dx.
\end{aligned}$$

This estimate is valid for any choice of the function η; therefore, letting $\{\eta=1\}$ invade the whole of B_r, we deduce that the term $\int_{B_r}\chi_{\{\eta<1\}}(x)\,dx$ can be made as small as we wish. As a consequence,

$$0\ \leqslant\ \int_{B_r}\big(|\nabla v_0|^2-|\nabla u_0|^2\big) + \lambda_2 Q(0)\,\Phi_0'(\Xi_0)\int_{B_r}\chi_{\{v_0>0\}}(x)\,dx - \lambda_2 Q(0)\,\Phi_0'(\Xi_0)\int_{B_r}\chi_{\{u_0>0\}}(x)\,dx,$$

which establishes the desired minimality property for u_0. □

11. Partial regularity of the free boundary

Using the subharmonicity property of the minimizers and their Lipschitz regularity (recall Lemma 2.2 and Theorem 1.3), we are now in the position to exploit standard techniques from geometric measure theory and conclude a partial regularity of the free boundary, as claimed in Theorem 1.6:

Proof of Theorem 1.6. The proof of (i) in Theorem 1.6 follows by a standard integration by parts (combined with the subharmonicity property of Lemma 2.2, see e.g. the proof of formula (8.1) in [9] for details).

To prove (ii) in Theorem 1.6, we argue by contradiction and we suppose that there exist $r_j \searrow 0$ and $x_j \in \Gamma$ such that $B_{2r_j}(x_j) \subset \Omega$ and

$$\int_{B_{r_j}(x_j)} \Delta u^+ \leqslant \frac{r_j^{n-1}}{j}. \tag{11.1}$$

Up to subsequences, we may suppose that $x_j \to \bar{x} \in \overline{D} \subset \Omega$. We define

$$v_j(x) := \frac{u(x_j + r_j x)}{r_j}.$$

Notice that, in the light of Corollary 5.3, for any $x, y \in \overline{B_1}$ we have

$$|v_j(x) - v_j(y)| \leqslant \frac{\big|u(x_j + r_j x) - u(x_j + r_j y)\big|}{r_j} \leqslant \frac{C\,|(x_j + r_j x) - (x_j + r_j y)|}{r_j} \leqslant C\,|x - y|.$$

So v_j (and hence v_j^+) is Lipschitz in $\overline{B_1}$ uniformly in j and thus we may suppose that v_j^+ converges to some $\bar{v}^+$ uniformly in $\overline{B_1}$.

Now, let $\phi \in C_0^\infty\big(B_1,\, [0, +\infty)\big)$. Notice that, by Lemma 2.2, we know that u is subharmonic, hence so is u^+ and $\Delta u^+ \geqslant 0$. Hence, recalling (11.1), we have

$$\begin{aligned}
\int_{B_1} \bar{v}^+ \Delta\phi &= \lim_{j\to+\infty} \int_{B_1} v_j^+ \Delta\phi = \lim_{j\to+\infty} r_j^{-1} \int_{B_1} u^+(x_j + r_j x)\, \Delta\phi(x)\, dx \\
&= \lim_{j\to+\infty} r_j \int_{B_1} \Delta u^+(x_j + r_j x)\, \phi(x)\, dx = \lim_{j\to+\infty} r_j^{1-n} \int_{B_{r_j}(x_j)} \Delta u^+(\xi)\, \phi\left(\frac{\xi - x_j}{r_j}\right) d\xi \\
&\leqslant \sup_{\mathbb{R}^n} \phi \lim_{j\to+\infty} r_j^{1-n} \int_{B_{r_j}(x_j)} \Delta u^+(\xi)\, d\xi \leqslant \sup_{\mathbb{R}^n} \phi \lim_{j\to+\infty} \frac{1}{j} = 0.
\end{aligned}$$

Accordingly, $\bar{v}^+$ is superharmonic in B_1. By construction, we also have that $\bar{v}^+ \geqslant 0$ and

$$\bar{v}^+(0) = \lim_{j\to+\infty} \frac{u(x_j)}{r_j} = \lim_{j\to+\infty} \frac{0}{r_j} = 0.$$

As a consequence

$$\bar{v}^+ \text{ vanishes identically in } B_1. \tag{11.2}$$

On the other hand, by (1.19),

$$c r_j^2 \leqslant \fint_{B_{r_j}(x_j)\cap\Omega_0^+} u^2 \leqslant \sup_{B_{r_j}(x_j)} u^2 \leqslant r_j^2 \sup_{B_1} v_j^2,$$

for some $c > 0$. So, simplifying r_j on both sides of the inequality and taking the limit in j, we find that

$$c \leqslant \sup_{B_1} \bar{v}^2.$$

This is in contradiction with (11.2) and so the proof of (ii) is complete.

Then, (1.20) follows from point (ii) and suitable geometric measure theory arguments (see e.g. the proof of Corollary 8.2 in [9] for full details). For the proof of (1.21), see e.g. the proof of Theorem B in [9]. □

12. Regularity of the free boundary

In this section we show that at flat points the free boundary is a regular smooth surface. In particular, in two spatial dimensions the free boundary is a continuously differentiable curve.

Our approach differs from the one in [1, 2] as we avoid using the flatness classes. Instead, we use the free boundary regularity theory for the viscosity solutions from [9].

12.1. **Viscosity solutions.** Recall the definitions of $\Omega^+(u)$ and $\Omega^-(u)$ given in the section with the notation.

If the free boundary is C^1 smooth, then

$$G(u_\nu^+, u_\nu^-) := (u_\nu^+)^2 - (u_\nu^-)^2 - \Lambda$$

is the flux balance across the free boundary, where u_ν^+ and u_ν^- are the normal derivatives in the inward direction to $\partial\Omega^+(u)$ and $\partial\Omega^-(u)$, respectively, and Λ is defined in (1.14).

With this notation, we give the definition of viscosity solution:

Definition 12.1. Let Ω be a bounded domain of $\mathbb{R}^n$ and let u be a continuous function in Ω. We say that u is a viscosity solution in Ω if

i) $\Delta u = 0$ in $\Omega^+(u)$ and $\Omega^-(u)$,
ii) along the free boundary Γ, u satisfies the free boundary condition, in the sense that:
   a) if at $x_0 \in \Gamma$ there exists a ball $B \subset \Omega^+(u)$ such that $x_0 \in \partial B$ and

$$u^+(x) \geqslant \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|), \text{ for } x \in B,$$

$$u^-(x) \leqslant \beta\langle x - x_0, \nu\rangle^- + o(|x - x_0|), \text{ for } x \in B^c,$$

   for some $\alpha > 0$ and $\beta \geqslant 0$, with equality along every non-tangential domain, then the free boundary condition is satisfied

$$G(\alpha, \beta) = 0,$$

   b) if at $x_0 \in \Gamma$ there exists a ball $B \subset \Omega^-(u)$ such that $x_0 \in \partial B$ and

$$u^-(x) \geqslant \beta\langle x - x_0, \nu\rangle^- + o(|x - x_0|), \text{ for } x \in B,$$

$$u^+(x) \leqslant \alpha\langle x - x_0, \nu\rangle^+ + o(|x - x_0|), \text{ for } x \in \partial B,$$

   for some $\alpha \geqslant 0$ and $\beta > 0$, with equality along every non-tangential domain, then

$$G(\alpha, \beta) = 0.$$

Lemma 12.2. *Let u be a minimizer in Ω for the functional J in* (1.8)*. Then, u is also a viscosity solution in the sense of Definition* 12.1*.*

Proof. See Lemma 11.17 in [7] or Theorem 4.2 in [9] for the proof. □

Notice that, if $x_0 \in \partial_{\mathrm{red}}\{u > 0\}$, then Γ is flat near x_0. Therefore, since a minimizer u is also a viscosity solution, according to Lemma 12.2, we can use the Harnack inequality approach to u, and obtain that Γ is $C^{1,\alpha}$ in some neighborhood of x_0.

We next show that in two dimensions the free boundary is a continuously differentiable curve.

12.2. **The case in which u^- is degenerate.** In this section we show that near the points $x_0 \in \Gamma$ where u^- is degenerate, the minimizer u behaves essentially as a solution to the one-phase problem. Recall that we say that u^- is degenerate at x_0 if

$$\liminf_{r\to 0} \frac{1}{r} \fint_{B_r(x_0)} u^- = 0. \tag{12.1}$$

We stress that u^+ (in contrast to u^-) is always nondegenerate, according to the following observation:

Lemma 12.3. *Let u be a minimizer in Ω for the functional J in* (1.8)*, with $0 \in \partial\{u > 0\}$. Assume that* (5.16) *and* (5.17) *are satisfied. Then*

$$\liminf_{r\to 0} \frac{1}{r} \fint_{B_r} u^+ > 0.$$

Proof. By the clean ball condition in Theorem 8.1, we have that $B_{c_1 r}(y_0) \subseteq B_r \cap \Omega^+(u)$, for a suitable point $y_0 \in B_r$ and a constant $c_1 > 0$.

Thus, from Lemma 5.4, we have that $u(y) \geqslant c_2 r$ for any $y \in B_{c_1 r/2}(y_0)$, for some $c_2 > 0$. As a consequence,

$$\int_{B_r} u^+ \geqslant \int_{B_{c_1 r/2}(y_0)} u^+ \geqslant c_2 r\, \mathcal{L}^n(B_{c_1 r/2}(y_0)),$$

which gives the desired result. □

Next, we show that if u^- is degenerate then so is the gradient of u^- in the following sense:

Lemma 12.4. *Let u be a minimizer in Ω for the functional J in* (1.8)*. Let $x_0 \in \partial\{u > 0\}$, and suppose that u^- is degenerate at x_0. Then*

$$\lim_{r \to 0} \sup_{x \in B_r(x_0)} |\nabla u^-(x)| = 0.$$

Proof. We argue by contradiction and we suppose that the conclusion of the lemma fails. Then, there exists a sequence $r_j \to 0$, as $j \to +\infty$, such that

$$\lim_{j \to +\infty} \frac{1}{r_j} \fint_{B_{r_j}(x_0)} u^- = 0 \tag{12.2}$$

$$\text{and} \qquad \lim_{j \to +\infty} \sup_{B_{r_j}(x_0)} |\nabla u^-| > 0. \tag{12.3}$$

Consider the scaled functions $u_j(x) := \frac{u(x_0 + r_j x)}{r_j}$. From Theorem 1.3 we obtain that $|\nabla u_j|$ is bounded uniformly in j in B_1. Then, up to a subsequence, we have that $u_j \to u_0$ as $j \to +\infty$ uniformly in B_1, for some function u_0. Moreover, by (12.2),

$$0 = \lim_{j \to +\infty} \frac{1}{r_j} \fint_{B_{r_j}(x_0)} u^- = \lim_{j \to +\infty} \fint_{B_1} u_j^- = \fint_{B_1} u_0^-.$$

This implies that

$$u_0^- = 0 \text{ in } B_1. \tag{12.4}$$

In addition, by (12.3) and Lemma 10.2,

$$0 < \lim_{j \to +\infty} \sup_{B_{r_j}(x_0)} |\nabla u^-| = \lim_{j \to +\infty} \sup_{B_1} |\nabla u_j^-| = \sup_{B_1} |\nabla u_0^-|,$$

which is in contradiction with (12.4). □

From Lemmata 12.3 and 12.4 it follows that, near a degenerate point $x_0 \in \Gamma$, u behaves almost like a minimizer of a one-phase functional. It is well-known that for the one-phase the gradient $|\nabla u|$ is upper semicontinuous. The aim of the next two lemmata is to establish this property of the gradient near degenerate points.

First we recall the Bernoulli constant

$$\Lambda(x_0) := \left[\lambda_2 \partial_{r_2} \Phi \left(\lambda_1 \int_\Omega Q\, \chi_{\{u<0\}},\ \lambda_2 \int_\Omega Q\, \chi_{\{u>0\}}\right) - \lambda_1 \partial_{r_1} \Phi \left(\lambda_1 \int_\Omega Q\, \chi_{\{u<0\}},\ \lambda_2 \int_\Omega Q\, \chi_{\{u>0\}}\right)\right] Q(x_0) \tag{12.5}$$

measuring the gradient jump across the free boundary. We observe that, in view of Lemma 6.1 and using the notation in (6.1), if x_0 is a smooth point of $\partial\{u > 0\}$, we know that

$$\left(\partial_\nu^+ u(x_0)\right)^2 - \left(\partial_\nu^- u(x_0)\right)^2 = \Lambda(x_0). \tag{12.6}$$

Furthermore, from Lemma 12.4 we should get that $(\partial_\nu u^+(x))^2 = \Lambda(x_0) + o(1)$ as $x \to x_0$. The next lemma makes this statement precise.

Lemma 12.5. *Let u be a minimizer in Ω for the functional J in* (1.8)*. Let $x_0 \in \partial\{u > 0\}$, and suppose that u^- is degenerate at x_0. Then,*

$$\limsup_{x \to x_0} |\nabla u(x)|^2 = \Lambda(x_0). \tag{12.7}$$

Proof. Let us denote by

$$\gamma := \limsup_{x\to x_0} |\nabla u(x)|. \tag{12.8}$$

By Lemma 12.4,

$$\gamma = \limsup_{x\to x_0} |\nabla u^+(x)|,$$

hence, in order to prove (12.7), one has to show that

$$\gamma^2 = \Lambda(x_0). \tag{12.9}$$

For this, we take a sequence $x_k \to x_0$ such that $x_k \in \{u > 0\}$ and $|\nabla u^+(x_k)| \to \gamma$ as $k \to +\infty$.

Let $\rho_k := \mathrm{dist}(x_k, \partial\{u > 0\})$ and let $y_k \in \partial\{u > 0\}$ such that $\rho_k = |x_k - y_k|$. Consider the blow-up sequence $u_k(x) := \frac{u(y_k+\rho_k x)}{\rho_k}$. From Proposition 10.1, up to a subsequence, we may assume that $u_k \to u_0$ as $k \to +\infty$ locally uniformly.

Without loss of generality we can also assume that

$$\frac{x_k - y_k}{\rho_k} \to -e_n, \quad \text{as } k \to +\infty,$$

where e_n is the unit direction of the x_n axis. Thus we have that

$$B_1(-e_n) \subseteq \{u_0 > 0\}. \tag{12.10}$$

This and Lemma 10.2 give that

$$\gamma = \lim_{k\to+\infty} |\nabla u^+(x_k)| = \lim_{k\to+\infty} \left|\nabla u_k^+\left(\frac{x_k - y_k}{\rho_k}\right)\right| = |\nabla u_0(-e_n)|. \tag{12.11}$$

From (12.10), we also obtain that

$$u_0 \text{ is harmonic in } B_1(-e_n), \tag{12.12}$$

thanks to Lemma 2.2 and Proposition 10.1.

We also observe that

$$|\nabla u_0| \leqslant \gamma \text{ in } B_1(-e_n). \tag{12.13}$$

Indeed, if $\bar{x} \in B_1(-e_n)$, we write $\bar{x} = -e_n + z$, with $|z| < 1$ and we set

$$z_k := y_k + \rho_k \bar{x} = y_k + \rho_k z - \rho_k e_n = \rho_k \left(\frac{y_k - x_k}{\rho_k} - e_n\right) + \rho_k z + x_k \to x_0,$$

as $k \to +\infty$. Thus, by (12.8),

$$\gamma = \limsup_{x\to x_0} |\nabla u(x)| \geqslant \lim_{k\to+\infty} |\nabla u(z_k)|. \tag{12.14}$$

On the other hand,

$$\nabla u_k(\bar{x}) = \nabla u(y_k + \rho_k \bar{x}) = \nabla u(z_k).$$

Hence, taking the limit as $k \to +\infty$ (and recalling (12.10) and Lemma 10.2), we see that

$$|\nabla u_0(\bar{x})| = \lim_{k\to+\infty} |\nabla u(z_k)|.$$

This and (12.14) imply (12.13), as desired.

We also remark that

$$\gamma > 0. \tag{12.15}$$

Indeed, if $\gamma = 0$, it follows from (12.13) that u_0 is constant in $B_1(-e_n)$. Thus, since

$$u_0(0) = \lim_{k\to+\infty} u_k(0) = \lim_{k\to+\infty} \frac{u(y_k)}{\rho_k} = 0, \tag{12.16}$$

we obtain that u_0 vanishes identically in $B_1(-e_n)$, in contradiction with (12.10), thus proving (12.15).

Now, we claim that

$$u_0(x) = -\nabla u_0(-e_n) \cdot x \text{ for any } x \in B_1(-e_n). \tag{12.17}$$

For this, we argue as follows: by (12.11) and (12.15), we can define

$$\ell := -\frac{\nabla u_0(-e_n)}{\gamma} = -\frac{\nabla u_0(-e_n)}{|\nabla u_0(-e_n)|}.$$

Then, by (12.11) and (12.13), we find that

$$\partial_\ell u_0(-e_n) = -\gamma \quad \text{and} \quad \partial_\ell u_0(x) \geqslant -\gamma \text{ in } B_1(-e_n). \tag{12.18}$$

Furthermore, in light of (12.12), we know that $\partial_\ell u_0$ is harmonic in $B_1(-e_n)$. This, (12.18) and the strong maximum principle imply that $\partial_\ell u_0 = -\gamma$ in $B_1(-e_n)$.

Then, we take a rotation $\mathcal{R}$ such that $e_1 = \mathcal{R}\ell$ and we define $v_0(x) := u_0(\mathcal{R}x)$. We have that

$$\partial_1 v_0(x) = \big(\mathcal{R}\nabla u_0(\mathcal{R}x)\big) \cdot e_1 = \nabla u_0(\mathcal{R}x)\big) \cdot \ell = -\gamma$$

for any x such that $\mathcal{R}x \in B_1(-e_n)$.

Consequently, for any x such that $\mathcal{R}x \in B_1(-e_n)$, we have that

$$v_0(x) = -\gamma x_1 + \tilde{v}(x_2, \dots, x_n), \tag{12.19}$$

for some $\tilde{v} : \mathbb{R}^{n-1} \to \mathbb{R}$. In particular,

$$|\nabla v_0|^2 = |\gamma|^2 + \sum_{i=2}^{n} |\partial_i \tilde{v}|^2. \tag{12.20}$$

On the other hand, by (12.13), for any x such that $\mathcal{R}x \in B_1(-e_n)$,

$$|\nabla v_0(x)|^2 = \big|\mathcal{R}\nabla u_0(\mathcal{R}x)\big|^2 \leqslant |\nabla u_0(\mathcal{R}x)|^2 \leqslant \gamma.$$

Then we insert this into (12.20) and we obtain that $\partial_i \tilde{v}$ vanishes identically for any $i \in \{2, \dots, n\}$, hence $\tilde{v}$ is constant, and (12.19) reduces to

$$v_0(x) = -\gamma x_1 + \tilde{c},$$

for some $\tilde{c} \in \mathbb{R}$.

Now, we recall (12.16) and we obtain that

$$0 = u_0(0) = v_0(0) = \tilde{c}.$$

As a consequence, we obtain that

$$v_0(x) = -\gamma x \cdot e_1 = -\gamma x \cdot \mathcal{R}^T \ell = -\gamma(\mathcal{R}x) \cdot \ell.$$

Hence

$$u_0(x) = v_0(\mathcal{R}^T x) = -\gamma x \cdot \ell,$$

thus completing the proof of (12.17).

Now, from (12.10) and (12.17), we deduce that $\ell = e_n$, and therefore

$$u_0(x) = -\gamma x_n \text{ in } B_1(-e_n). \tag{12.21}$$

We claim that, in fact,

$$u_0(x) = -\gamma x_n \text{ in } \{x_n < 0\}. \tag{12.22}$$

To check this, we recall (12.10) and we denote by $\mathcal{C}$ the connected component of $\{u_0 > 0\}$ that contains $B_1(-e_n)$. By Corollary 1.2, we know that u_0 is harmonic in $\mathcal{C}$. Hence, by (12.21) and the unique continuation principle, we obtain that $u_0(x) = -\gamma x_n$ in $\mathcal{C}$. As a consequence, since u_0 vanishes along $\partial\mathcal{C}$, we have that

$$\partial\mathcal{C} \subseteq \{-\gamma x_n = 0\} = \{x_n = 0\},$$

thanks to (12.15), and this establishes (12.22).

It remains to show that $\partial\{u_0 > 0\} = \{y_n = 0\}$. To see this it is enough to show that there exists $\delta > 0$ such that

$$u_0 = 0 \text{ in } \{x_n \in (0, \delta)\}. \tag{12.23}$$

Suppose that

$$s = \limsup_{\substack{y_n \downarrow 0,\\ y' \in \mathbb{R}^{n-1},\\ u_0(y',y_n)>0}} \frac{\partial u_0(y',y_n)}{\partial y_n}.$$

Note that u_0 is a minimizer of ACF functional, see Theorem 1.5. Taking a sequence $\frac{\partial u_0(y'_k,h_k)}{\partial y_n} \to 0$ as $h_k \to 0$ and using the same argument above it follows that the second blow of u_0, which we call u_{00}, with respect to the balls $B_{h_k}(y'_k,0)$ is of the form $u_{00} = sy_n$, with $y_n > 0$. This is a contradiction, since the zero set of the minimizers of ACF functional has nontrivial measure, see also Theorem 9.2. Thus it follows that $s = 0$ and consequently we have that $u_0 = 0$ in some strip $\{0 < y_n < \delta\}$, for a suitable $\delta > 0$. This establishes (12.23).

Now, in light of (12.6), (12.22), and (12.23), we have that

$$\Lambda(x_0) = \big(\partial^+_\nu u_0(0)\big)^2 - \big(\partial^-_\nu u_0(0)\big)^2 = \gamma^2 - 0,$$

which proves (12.9), as desired. □

Having established the upper semicontinuity of $|\nabla u|$ at the free boundary points where u^- is degenerate, we next establish an estimate for the upper modulus of continuity. We recall that we are working under the assumptions in (1.15).

Lemma 12.6. *Assume that u is a minimizer in Ω for the functional J in* (1.8)*. Let*

$$S := \sup_{r\in[0,\lambda_2 Q_2 \mathcal{L}^n(\Omega)]} \Phi'_0(r) < +\infty.$$

Let $x_0 \in \partial\{u > 0\}$, and suppose that u^- is degenerate at x_0. Then, there is $R > 0$ and $\alpha > 0$ such that

$$\sup_{B_r(x_0)} |\nabla u|^2 \leqslant \Lambda(x_0) + C\left(\frac{r}{R}\right)^\alpha + o(1), \quad \textit{as } r \searrow 0,$$

where $o(1) = \sup_{B_r(x_0)} |\nabla u^-|^2$.

Proof. We adapt a method from the proof of Theorem 4.1 in [3]. By Corollary 1.2, we have that $\Delta|\nabla u|^2 = 2\sum_{kl} u^2_{kl} \geqslant 0$ in $\{u > 0\}$. Thus $w := |\nabla u|^2$ is subharmonic in $\{u > 0\}$. Hence, it follows that

$$U_\varepsilon := (w - \Lambda^2(x_0) - \varepsilon)^+ \text{ is subharmonic in } \{u > 0\}. \tag{12.24}$$

From Lemma 12.5, we know that $U_\varepsilon = 0$ on $\partial\{u > 0\}$. So, we extend U_ε into $\{u \leqslant 0\}$ by zero and we deduce from (12.24) and the fact that $U_\varepsilon \geqslant 0$ that U_ε is subharmonic in the whole of Ω.

For any $r > 0$ (to be taken small in the sequel), we set

$$h_\varepsilon(r) := \sup_{B_r(x_0)} U_\varepsilon.$$

Then, we have that

$$h_\varepsilon(r) - U_\varepsilon \text{ is superharmonic in } B_r(x_0). \tag{12.25}$$

By construction, we have that

$$\begin{aligned} & h_\varepsilon(r) - U_\varepsilon \geqslant 0 \quad \text{in } B_r(x_0),\\ \text{and} \qquad & h_\varepsilon(r) - U_\varepsilon = h_\varepsilon(r) \quad \text{in } B_r(x_0) \cap \{u \leqslant 0\}. \end{aligned}$$

Accordingly, for any $p \geqslant 1$,

$$\begin{aligned} \|h_\varepsilon(r) - U_\varepsilon\|^p_{L^p(B_r(x_0))} &\geqslant \int_{B_r(x_0)\cap\{u\leqslant 0\}} (h_\varepsilon(r) - U_\varepsilon)^p = \int_{B_r(x_0)\cap\{u\leqslant 0\}} (h_\varepsilon(r))^p \\ &= (h_\varepsilon(r))^p\, \mathcal{L}^n(B_r(x_0) \cap \{u \leqslant 0\}). \end{aligned} \tag{12.26}$$

Now, by density estimates (recall Theorem 9.2), we have that $\mathcal{L}^n(B_r(x_0) \cap \{u \leqslant 0\}) \geqslant cr^n$, for some $c > 0$, and so we deduce from (12.26) that

$$\|h_\varepsilon(r) - U_\varepsilon\|^p_{L^p(B_r(x_0))} \geqslant c\,(h_\varepsilon(r))^p\, r^n. \tag{12.27}$$

Also, from (12.25) and the weak Harnack inequality for superharmonic functions, it follows that, for any $1 \leqslant p < \frac{n}{n-2}$,

$$\inf_{B_{r/2}(x_0)} [h_\varepsilon(r) - U_\varepsilon] \geqslant C r^{-\frac{n}{p}} \|h_\varepsilon(r) - U_\varepsilon\|_{L^p(B_r(x_0))}.$$

This and (12.27) imply that

$$\inf_{B_{r/2}(x_0)} [h_\varepsilon(r) - U_\varepsilon] \geqslant c h_\varepsilon(r),$$

for some $c \in (0,1)$, and therefore

$$(1-c)\, h_\varepsilon(r) \geqslant U_\varepsilon(x), \tag{12.28}$$

for any $x \in B_r(x_0)$. Thus, let us define

$$h_0(r) := \sup_{B_r(x_0)} (w - \Lambda(x_0))^+.$$

By construction, $h_0(r) \geqslant h_\varepsilon(r)$, hence we deduce from (12.28) that

$$(1-c)\, h_0(r) \geqslant U_\varepsilon(x) = (w(x) - \Lambda(x_0) - \varepsilon)^+$$

for any $x \in B_{r/2}(x_0)$. Thus, sending $\varepsilon \searrow 0$, we find that

$$(1-c)\, h_0(r) \geqslant (w(x) - \Lambda(x_0))^+$$

for any $x \in B_{r/2}(x_0)$, that is

$$(1-c)\, h_0(r) \geqslant \sup_{x \in B_{r/2}(x_0)} (w(x) - \Lambda(x_0))^+ = h_0\left(\frac{r}{2}\right).$$

By iterating this inequality, it follows that, for small $s > 0$, we have from Lemma 8.23 in [13] that $h_0(s) \leqslant C\left(\frac{s}{R}\right)^\alpha$, for some $C > 0$ and $\alpha \in (0,1)$. From this, the desired result follows. $\square$

Now we are ready to show that in two dimensions the free boundary is a continuously differentiable curve. To do so we only need to show that the free boundary is flat at each point. Then the result will follow from the viscosity regularity theory of Caffarelli [5, 6] and Lemma 12.2.

We begin with showing that at the points where u^- is degenerate the free boundary is indeed flat. The case of nondegenerate u^- will be studied in the next subsection.

Lemma 12.7. *Let $n = 2$ and suppose that* (9.3) *holds true. Let u be a minimizer in Ω for the functional J in* (1.8)*. Let $x_0 \in \partial\{u > 0\}$, and suppose that u^- is degenerate at x_0. Let also Λ be as in* (12.5) *and Q be continuous. Then,*

$$\lim_{r \to 0} \fint_{B_r(x_0) \cap \{u > 0\}} \max\{\Lambda(x_0) - |\nabla u|^2, 0\} = 0. \tag{12.29}$$

Proof. This type of result was proved in Theorem 6.6 of [1] in the case $\Phi_0(r) := r$ and $u \geqslant 0$. We need to adapt their strategy to the case under consideration. To this aim, for any $\varepsilon > 0$, let

$$u_\varepsilon := \max\{u - \varepsilon\zeta,\, 0\} - u^-,$$

where $\zeta \in C_0^\infty(B_\rho(x_0),\, [0,1])$. Notice that we can also write $u_\varepsilon = u - \min\{u, \varepsilon\zeta\} - u^-$.

Since $u_\varepsilon = u$ on $\Omega \setminus B_\rho(x_0)$, from the minimality of u, it follows that $J[u] \leqslant J[u_\varepsilon]$. Also, the support of Δu lies on the free boundary, where u vanishes. Recalling also Lemma 12.4, we see that $|\nabla u^-|^2 \leqslant \sigma(\rho) \to 0$ as $\rho \to 0$, and therefore

$$\begin{aligned}
&\Phi_0\left(\mathcal{M}_2(u)\right) - \Phi_0\left(\mathcal{M}_2(u - \min\{u, \varepsilon\zeta\} - u^-)\right) \\
\leqslant\ & \int_\Omega |\nabla(u - \min\{u, \varepsilon\zeta\} - u^-)|^2 - \int_\Omega |\nabla u|^2 \\
=\ & \int_{B_\rho(x_0)} |\nabla(\min\{u, \varepsilon\zeta\} + u^-)|^2 - 2\int_{B_\rho(x_0)} \nabla u \cdot \nabla(\min\{u, \varepsilon\zeta\} + u^-) \\
=\ & \int_{\{u>\varepsilon\zeta\}\cap B_\rho(x_0)} |\nabla(\varepsilon\zeta + u^-)|^2 + \int_{\{u\leqslant\varepsilon\zeta\}\cap B_\rho(x_0)} |\nabla u^+|^2 \\
& -2\int_{B_\rho(x_0)} \operatorname{div}\left((\min\{u, \varepsilon\zeta\} + u^-)\nabla u\right) + 2\int_{B_\rho(x_0)} \Delta u\,(\min\{u, \varepsilon\zeta\} + u^-) \\
\leqslant\ & C\int_{\{u>\varepsilon\zeta\}\cap B_\rho(x_0)} \left(\varepsilon^2|\nabla\zeta|^2 + |\nabla u^-|^2\right) + \int_{\{u\leqslant\varepsilon\zeta\}\cap B_\rho(x_0)} |\nabla u|^2 - 2\int_{\partial B_\rho(x_0)} (\min\{u, \varepsilon\zeta\} + u^-)\partial_\nu u \\
\leqslant\ & C\varepsilon^2 \int_{\{u>\varepsilon\zeta\}\cap B_\rho(x_0)} |\nabla\zeta|^2 + C\rho^2 \sup_{B_\rho(x_0)} |\nabla u^-|^2 + \int_{\{0<u\leqslant\varepsilon\zeta\}\cap B_\rho(x_0)} |\nabla u|^2 - 2\int_{\{u\leqslant 0\}\cap\partial B_\rho(x_0)} (u + u^-)\partial_\nu u \\
\leqslant\ & C\varepsilon^2 \int_{\{u>\varepsilon\zeta\}\cap B_\rho(x_0)} |\nabla\zeta|^2 + C\rho^2\sigma(\rho) + \int_{\{0<u\leqslant\varepsilon\zeta\}\cap B_\rho(x_0)} |\nabla u|^2.
\end{aligned}$$

Also, since Φ_0 is concave (recall (1.15)),

$$\begin{aligned}
&\Phi_0\left(\mathcal{M}_2(u - \min\{u, \varepsilon\zeta\} - u^-)\right) - \Phi_0\left(\mathcal{M}_2(u)\right) \\
\leqslant\ & \Phi_0'\left(\mathcal{M}_2(u)\right)\Big(\mathcal{M}_2(u - \min\{u, \varepsilon\zeta\} - u^-) - \mathcal{M}_2(u)\Big) \\
=\ & \Phi_0'\left(\mathcal{M}_2(u)\right)\,\lambda_2 \int_\Omega Q(x)\Big(\chi_{\{u-\min\{u,\varepsilon\zeta\}-u^->0\}} - \chi_{\{u>0\}}\Big) \\
=\ & -\Phi_0'\left(\mathcal{M}_2(u)\right)\,\lambda_2 \int_{B_\rho(x_0)} Q(x)\,\chi_{\{0<u\leqslant\min\{u,\varepsilon\zeta\}+u^-\}} \\
\leqslant\ & -\Phi_0'\left(\mathcal{M}_2(u)\right)\,\lambda_2 \int_{\{0<u\leqslant\varepsilon\zeta\}\cap B_\rho(x_0)} Q(x).
\end{aligned}$$

From these observations, we find that

$$\int_{\{0<u\leqslant\varepsilon\zeta\}\cap B_\rho(x_0)} \Big(\Phi_0'\left(\mathcal{M}_2(u)\right)\,\lambda_2\, Q(x) - |\nabla u|^2\Big) \leqslant C\varepsilon^2 \int_{\{u>\varepsilon\zeta\}\cap B_\rho(x_0)} |\nabla\zeta|^2 + C\rho^2\sigma(\rho). \tag{12.30}$$

In view of (1.6),

$$\Phi_0'(r)\,\lambda_2 = -\lambda_1\partial_{r_1}\Phi\Big(\lambda_1\left(\lambda_\Omega - \lambda_2^{-1}r\right),\, r\Big) + \lambda_2\partial_{r_2}\Phi\Big(\lambda_1\left(\lambda_\Omega - \lambda_2^{-1}r\right),\, r\Big),$$

and so

$$\Phi_0'\left(\mathcal{M}_2(u)\right)\,\lambda_2\, Q(x) = \Lambda(x). \tag{12.31}$$

Now we fix $R > \rho > r$ and we choose $\varepsilon := Lr$, where $L > 0$ is the Lipschitz constant of u in $B_R(x_0)$ (recall Theorem 1.3). We take

$$\zeta(x) = \begin{cases} 0 & x \in \mathbb{R}^2 \setminus B_\rho(x_0), \\ \dfrac{\log(\rho/|x - x_0|)}{\log(\rho/r)} & x \in B_\rho(x_0) \setminus B_r(x_0), \\ 1 & x \in B_r(x_0). \end{cases}$$

We observe that, in $B_r(x_0)$, it holds that $u \leqslant Lr = \varepsilon = \varepsilon\zeta$. From this, (12.30) and (12.31) we infer that

$$\begin{aligned}
&\int\limits_{\{u>0\}\cap B_r(x_0)} \Big(\Lambda(x) - |\nabla u|^2\Big) + \int\limits_{\{0<u\leqslant rL\zeta\}\cap(B_\rho(x_0)\setminus B_r(x_0))} \Big(\Lambda(x) - |\nabla u|^2\Big) \\
\leqslant\ & CL^2r^2 \int\limits_{\{u>\varepsilon\zeta\}\cap B_\rho(x_0)} |\nabla\zeta|^2 + C\rho^2\sigma(\rho),
\end{aligned}$$

or equivalently

$$
\begin{aligned}
&\int\limits_{\{u>0\}\cap B_r(x_0)} \Big(\Lambda(x)-|\nabla u|^2\Big)^+ \\
\leqslant\ & \int\limits_{\{u>0\}\cap(B_\rho(x_0)\setminus B_r(x_0))} \Big(|\nabla u|^2-\Lambda(x)\Big) + \int\limits_{\{u>0\}\cap B_r(x_0)} \Big(\Lambda(x)-|\nabla u|^2\Big)^- \\
&\quad + CL^2r^2 \int\limits_{\{u>\varepsilon\zeta\}\cap B_\rho(x_0)} |\nabla\zeta|^2 + C\rho^2\sigma(\rho) \\
\leqslant\ & \int\limits_{\{u>0\}\cap(B_\rho(x_0)\setminus B_r(x_0))} \Big(|\nabla u|^2-\Lambda(x)\Big) + \int\limits_{\{u>0\}\cap B_r(x_0)\cap\{|\nabla u|^2>\Lambda(x)\}} \Big(|\nabla u|^2-\Lambda(x)\Big) \\
&\quad + CL^2r^2 \int\limits_{\{u>\varepsilon\zeta\}\cap B_\rho(x_0)} |\nabla\zeta|^2 + C\rho^2\sigma(\rho) \\
\leqslant\ & C\rho^2\left(\left(\frac{\rho}{R}\right)^\alpha + C\sigma(\rho)\right) + \frac{CL^2r^2}{\log\left(\frac{\rho}{r}\right)} + C\rho^2\sigma(\rho),
\end{aligned}
$$

where we have also used Lemma 12.6. After dividing both sides of the last identity by r^2 we get

$$
\frac{1}{r^2}\int_{\{u>0\}\cap B_r(x_0)}\Big(\Lambda(x)-|\nabla u|^2\Big)^+ \leqslant C\left(\frac{\rho}{r}\right)^2\left(\frac{\rho}{R}\right)^\alpha + \frac{CL^2}{\log\left(\frac{\rho}{r}\right)} + C\left(\frac{\rho}{r}\right)^2\sigma(\rho). \tag{12.32}
$$

Now we choose $r := \rho\sigma^\beta(\rho)$, for some $\beta\in(0,\frac{1}{2})$, and $R = \frac{\rho}{\sigma^{\frac{3\beta}{\alpha}}(\rho)}$ and we see that

$$
C\left(\frac{\rho}{r}\right)^2\left(\frac{\rho}{R}\right)^\alpha + \frac{CL^2}{\log\left(\frac{\rho}{r}\right)} + \left(\frac{\rho}{r}\right)^2\sigma(\rho) \leqslant C\sigma^\beta(\rho) + \frac{CL^2}{\beta\log\frac{1}{\sigma(\rho)}} + C\sigma^{1-2\beta}(\rho).
$$

Finally choose $\beta := \frac{1}{3}$ to infer that

$$
\frac{1}{r^2}\int_{\{u>0\}\cap B_r(x_0)}\Big(\Lambda(x)-|\nabla u|^2\Big)^+ \leqslant C\sigma^{\frac{1}{3}}(\rho) + \frac{3CL^2}{\log\frac{1}{\sigma(\rho)}} \to 0 \quad \text{as} \quad \rho\to 0,
$$

and the desired result follows. □

12.3. **The case in which u^- is nondegenerate.** If u^- is nondegenerate then the Alt-Caffarelli-Friedman functional

$$
\phi(r,x_0,u) = \frac{1}{r^4}\int_{B_r(x_0)}\frac{|\nabla u^+|^2}{|x-x_0|^2}dx\int_{B_r(x_0)}\frac{|\nabla u^-|^2}{|x-x_0|^2}dx \tag{12.33}
$$

has positive limit and therefore from Theorem 7.4 (i) in [2] when $n=2$ the blow-up u_0 must be a two-plane solution.

Lemma 12.8. *Let u be a minimizer in Ω for the functional J in* (1.8)*, with $0\in\partial\{u>0\}$. Assume that* (5.16) *and* (5.17) *are satisfied.*

Let $\{r_j\}_{j\in\mathbb{N}}$ be a sequence of positive numbers such that $r_j\searrow 0$ as $j\to+\infty$. Let $x_0\in\partial\{u>0\}$ and set $u_j(x) := \frac{u(x_0+r_jx)}{r_j}$ such that $u_j\to u_0$ as $j\to+\infty$ for some subsequence, still denoted by $\{r_j\}$. If $n=2$ and u^- is nondegenerate at x_0 then

$$
\lim_{r_j\to 0}\phi(r_j,x_0,u) = \gamma > 0
$$

and u_0 is a two-plane solution, namely

$$
u_0(x) = \mu_1(x\cdot\ell)^+ - \mu_2(x\cdot\ell)^-
$$

for some unit direction ℓ and positive constants μ_1,μ_2.

Proof. From the scale invariance of the Alt-Caffarelli-Friedman functional we have that

$$
\phi(r_js,x_0,u) = \phi(s,0,u_j).
$$

Since, by Lemma 12.3, we have that u^+ is nondegenerate, and by assumption so is u^- at x_0, then it follows that the limit

$$
\lim_{r_j\to 0}\phi(sr_j,x_0,u)
$$

exists and is independent of $s > 0$, because ϕ is monotone and bounded thanks to Lipschitz continuity of u. Therefore we have that

$$\phi(s, 0, u_0) = \gamma > 0, \qquad \forall s > 0,$$

which implies that u_0 must be a homogeneous function of degree 1 (by the nondegeneracy of u_0^+ and the Lipschitz regularity of u, recall Theorem 1.4 and Corollary 5.3). Applying Lemma 6.6 in [2], the desired result follows. ☐

Summarizing Lemmata 12.7 and 12.8, we obtain the result in Theorem 1.7:

Proof of Theorem 1.7. By Lemma 12.2 we know that u is a viscosity solution. It follows from Lemmata 12.7 and 12.8 that the free boundary $\partial\{u > 0\}$ is flat at each point. Hence, the proof of the theorem follows from the regularity theory of Caffarelli developed for the viscosity solutions [5, 6]. See also Proposition 6.1 in [9]. ☐

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